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Astronomy.

A Sphere is a solid body formed by the complete revolution of a semicircle round its diameter.

A great circle of a sphere is the intersection of the spherical surface with a plane passing through its center.

The center of the semicircle becomes the center of the sphere and its diameter the axis of the sphere.

The poles of a spere are the intersections of its axis with its surface.

A great circle of a sphere is the intersection of the spherical surface with a plane passing through its center.

A small circle of a sphere is the intersection of the surface with a plane not passing through its center.

The poles of a great circle are the intersections of the spherical surface with a right line passing through center of the sphere at right angle to the plane of the great circle; and that perpendicular is called its axis.

The secondaries to a great circle are those great circles which pass through its poles and whose planes are perpendicular to its plane.

Parallel circles to a great circle are small circles of a sphere whose planes are parallel to its plane

As all the heavenly bodies are so far removed from us that our eyes cannot perceive their difference of distance; we may suppose them all projected upon a concave sphere concentric with the earth. This is called the Concave Sphere of the heavens.

The Equator is a great circle of that sphere in whose plane the

is parallel to the circles of the ~~the earths proposed~~ diurnal motion of the fixt stars —

The axis of the earth ~~the celestial Equator~~ provided marks the poles of the heavens and of the celestial Equator.

The Terrestrial Equator or the Equinoctial line is the intersection of the earth's surface with the plane of the celestial Equator. The poles of the Earth are the intersections of the Axis of the Equator with the Earths surface

The secondaries to the Terrestial equator are called Meridians.

The Latitude of a place upon the earth is its Distance from the equator measured upon the meridian passing through it

The Longitude of a place is the arc of the equator intercepted between the ~~point in which~~ the meridian passing through the place and a given meridian.

The Ecliptic is ~~that~~ great circle of the sphere in whose plane the {sun / earth performs its annual} appears to move annually. motion.

The intersections of the Ecliptic and Equator are called the Equinoxes.

The Solstices are two points upon the ecliptic 90 degrees distant from the Equinoxes.

The Parallels to the Equator passing through the Solstices are called the tropics.

* The Declination of a heavenly body is its distance from the Equator measured upon a secondary to the Equator.

The Ecliptic is divided into twelve equal parts called signs, aries taurus &c beginning from an equinox.

The right ascension of a heavenly body is an arc of the Equator intercepted between the first of aries and the secondary to the equator passing through it.

The Latitude of a heavenly body is its distance from the ecliptic measured upon the secondary to the ecliptic passing through it.

The Longitude of a heavenly body is an arc of the Ecliptic intercepted between the first of Aries and the secondary to the ecliptic passing through it.

The Sensible Horizon is a plane touching the earth at the place of the observer.

The Rational Horizon is a great circle of the sphere whose plane is parallel to the sensible horizon

The Axis of the horizon passes through the place of the observer and marks its poles the Zenith above and the Nadir below,

The secondaries to the horizon are called verticals 2

The Meridian of is a common secondary to the equator and horizon; its intersections with the horizon are called South and North;

The Prime vertical is a secondary to the horizon at right angles to the meridian; its intersections with the horizon mark the East and West.

The Altitude of a heavenly body is its distance from the horizon measured upon the vertical passing through it.

The Azimuth of a star is the arc of the horizon intercepted between the meridian and a vertical passing through it.

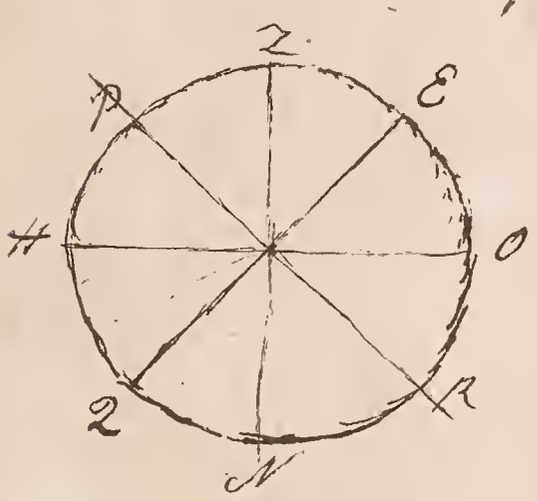
The Amplitude is the arc of the horizon intercepted between the prime vertical and the point where the star rises.

* The Polar circles are two parallels to the equator 23½ degrees distant from the poles.

To find the elevation of the Pole.

All the ~~heavenly bodies~~ fixt stars appear to ~~re~~volve in parallel circles round a common axis, once in ~~four and twenty hours~~ 23 hours & 56 minutes; some are ~~longer~~ the same time above the horizon as below it, some more and some less, some never appear, others never set; to find the axis of this motion, which produced marks the pole in the heavens, take one of those stars which never set, observe its greatest and least altitudes bisect the distance between them and add it to the least altitude, the sum will be the elevation of the pole.

The altitude of the Equator is the complement of the elevation of the Pole.



Let P be the pole HO the horizon and EQ the Equator; PH will be the elevation of the pole and EO the altitude of the equator. The arc HZO is equal to 180 degrees and PE equal to 90, their difference PH + EO equal to 90 degrees and consequently E is the complement of PH.

The Earth is known to be of a spherical form from the different elevations of the pole and Equator in different places; from the earth's shadow in eclipses of the moon which always appears round; from the convexity of the sea where the top of an object, at a distance, is seen and the bottom covered; and from the voyages that have been made round it.

The magnitude of the earth (supposing it a perfect sphere) is found by observing the elevation of the pole at a certain place upon the earth; then going due North till it rises one South till it falls a degree the Distance between the two observations will be a degree of the earths circumference whi

ing found, the magnitude of the earth is easily determined; by this method a degree has been found equal to 25 and the whole circumference or 360 x 25 to 9000 French leagues.

The ~~longitude~~ latitude of a place is equal to the elevation of the pole; because the latitude of a place is its distance from the equator on the earth or of its zenith from the equator in the heavens which is equal to the elevation of the pole

As the sun moves round the earth once in 24 hours, the longitude of places upon the earth may be measured by time; Divide ~~it~~ the whole equator into parts of 15 degrees each, then as the Sun's diurnal motion is uniform it is evident that the sun will pass each of these divisions in an hours time (as 24 x 15 = 360). The meridians that mark these divisions upon the equator are called horary circle,

When the sun is in the meridian, or when it is twelve o'clock at a certain place; fifteen degrees to the east it is one o'clock, and fifteen degrees to the west, eleven; so that if a person were to travel 15 degrees from west to east he would see the sun in the meridian an hour sooner, and if he went round the world in the same direction, his absence would appear to him 24 hours ~~longer~~ shorter than it really was; The contrary would happen were he to travel from East to west.

The distance between the meridians of two places on the earth, or their difference of longitude is found by knowing exactly the time of day when a common phenomenon such as an eclipse, appears at the different places.

The Sun's motion in the Ecliptic

The Sun is known to have a motion different from that of its diurnal revolution by comparing it with the fixt stars. If a star is observed several days successively after sun-set, it will appear every day nearer & nearer to the sun till its light is lost in the rays of the sun; and this effect is produced by the motion of the sun towards the star not of the star towards the sun, because the fixt stars rise at the same point and set at the same points of the visible horizon every day, whereas the sun every day changes its point of rising and setting; and because the fixt stars have no relative motion with respect to each other.

By this observation it is found that the sun has a motion nearly of a degree in four and twenty hours, in a direction contrary to that of its diurnal revolution; and that at the end of 365 days and about six hours it returns to the same place with respect to the fixt stars. The path of this motion is found to be a great circle of the sphere (that is the motion is always made in a plane passing through the center of that sphere, which being projected upon the concave surface, appears as a great circle) this circle is called the Ecliptic.

The Obliquity of the Ecliptic.

The circle of the ecliptic is different from that of the equator; for were the sun to move in the equator it would pass the meridian every day at the same altitude; but in summer it rises 23½ degrees above the equator & in winter falls 23½ degrees beneath so that these two circles

are inclined to one another at an angle of 23½ degrees. The intersections of these circles are called equinoxes, because when the sun is in the equator it is twelve hours above and twelve hours beneath the horizon.

~~The Solstices~~ The Zones.

The surface of the earth is divided into five parts; the torrid, the two temperate and the two frigid zones; the torrid is the space between the two tropics, the temperate between the tropics and the polar circles, and the frigid ~~contained~~ within ~~the~~ the polar circles. As every point in the torrid zone has the sun in its zenith ~~once or~~ twice in the year, its inhabitants have been called Ascii from the shadow being entirely lost; they are likewise called Amphiscii from their having the meridian shadow ~~on one side~~ sometimes to the south, sometimes to the north.

The inhabitants of the temperate zones are called Heteroscii; because the sun, never passing beyond the tropics, throws the meridian shadows of the two temperate zones different ways. The inhabitants of the frigid zones are called Periscii; because within these circles the sun remaining more than four and twenty hours above the horizon at least once in the year the shadows will move round in every direction.

Situations of the Sphere

A place under the equator is said to be in a right sphere because its horizon is at right angles to the equator and its parallels; a place under one of the poles is said to be in a parallel sphere because its horizon is parallel to the equator; a place between the equator and poles is in an oblique sphere because

its horizon makes an angle with the equator; in the right sphere, as the horizon cuts all the parallels into two equal parts; all the stars will be twelve hours above and the same time beneath the horizon and the length of the day will always be equal to that of the night; in the parallel sphere, as the horizon ~~is parallel to the parallels~~ coincides with the equator, all the stars on one side of the equator will be seen and will never set and those on the other side will never rise, and the sun will be one half of the year above the horizon and the other beneath it; In the obligue sphere, the horizon meets the parallels to the equator obliquely it will cut them unequally and all the stars, those in the equator excepted, will be longer on one side of the horizon ~~equator~~ than on the other for this reason the days and nights are of unequal lengths except ~~at the equinoxes~~ when the sun is in the equator

As the heat of the Sun depends upon the perpendicular direction of its rays ~~and the length of its day above the horizon~~, the nearer a place is situated to either of the poles, the colder will be its climate and the greater the variety of its seasons. The

Antipodes are those that inhabit the other extremity of that diameter of the earth which passes through the place of the observer, as they are placed on opposite meridian and upon a circle of ~~contrary~~ latitude equal and contrary latitude their night will correspond to our day and their winter to our summer &c; The Antaeci ~~on~~ have the same meridian with us and are situated upon circles of equal and contrary latitude;

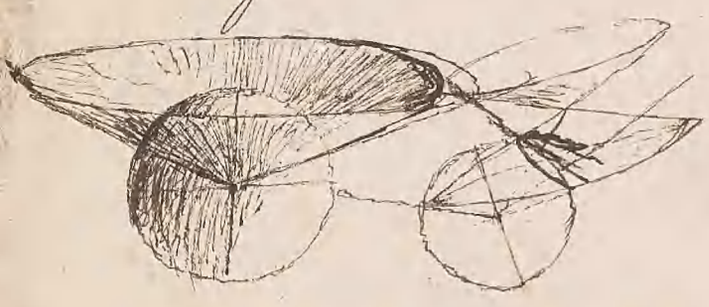
they have ~~mid-day~~ noon and mid night at the same time with us, but the seasons are contrary; the length of their night corresponds ~~with~~ to that of our day. The Periaci are situated upon the same circle of latitude with the observer but upon a contrary meridian; they have the same seasons and the same length of day and night with us; but to their ~~mid-day~~ noon corresponds ~~with~~ to our mid-night &c.

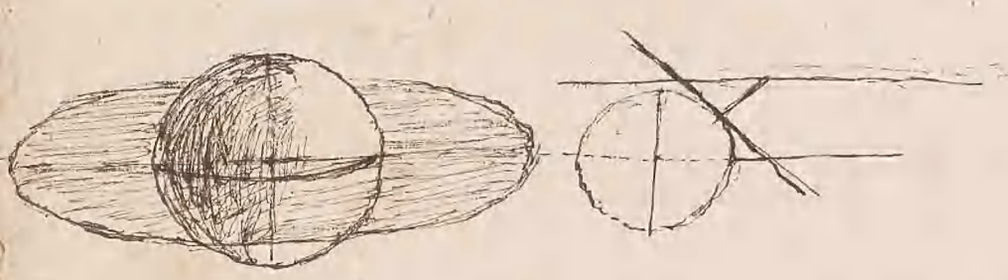
To draw a meridian line.

As the sun every day describes nearly a parallel to the Equator, the meridian of a place will pass through the point of its highest greatest elevation; this point it is to be found by taking two points of equal altitude in the path of the sun and bisecting the arc between them. To trace a meridian line upon a horizontal plane, at a point

A erect a style at right angles to the plane, observe the shadow of the top of the style before twelve O'clock, as at B, describe a circle with A as its center and A B its radius, observe the point in which the shadow of the ~~sun~~ crosses the circle in the forenoon as at C, bisect the arc C B in D and join A D, A D will be the direction of the meridian line required; for greater exactness several points in D may be found by taking points as B and describing circles from the center A. The path of the shadow of the top of the style upon the horizontal plane is universally one of the conic sections, which varies and changes with the place of the observer and the place of the sun in the ecliptic; at the pole it is a circle; elsewhere, when the sun is 24 hours above the horizon it is an ellipse; where it touches the horizon at its lowest point, a parabola; and in every other situation till it is not seen, an hyperbola; when the Sun is in the

in the equator, the locus of the shadow of ☉, in every part of the globe, is a right line. For, where the sun is not in the equator the locus of the shadow is formed by the intersection of the plane of the horizon with the surface of a cone, formed by the revolution of the sun's rays round the earth; because the direction of the rays is inclined to the axis





of motion ~~of the earth's motion~~; but when the direction of the rays is perpendicular to the axis, that is, when the sun is in the equator, ~~the vertical angle~~ of the cone becomes equal to two right angles, ~~and~~ its surface coincides with the plane of the equator; and its intersection with the plane of the horizon, which is the locus of the shadow, becomes a straight line; as the sun is so far distant from the earth that its rays may be considered as falling parallel upon its surface every point in it may be considered as in the plane of the ecliptic equator path of the sun's motion, which is confined to that plane, will be projected into a right line.

: I. A I E

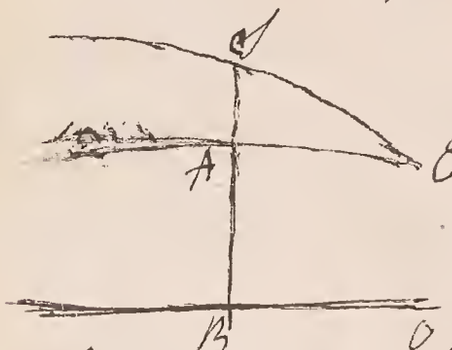
*R : Cos. E :: I. IE : I. A E the sun's right ascension. Cor. A I : R :: Cos. E : ☊

A I E is the complement to the angle contained between the circles of the sun's latitude and declination, called the angle of position

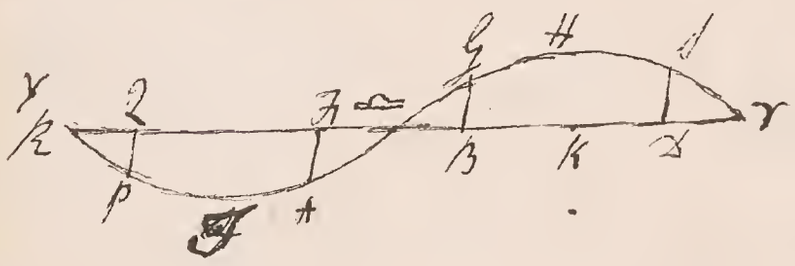
The Sun's motion & inequalities.

Knowing the places of the equator, Ecliptic & ~~equinoxes~~ equinoxes, the sun's place in the ecliptic ~~may be found~~ or its longitude may be found by observing its altitude in passing the meridian.

Let $\underline{\text{A}}\,\underline{\text{E}}$ be the equator & $\underline{\text{1}}\,\underline{\text{E}}$ the ecliptic meeting



in the equinox $\underline{\text{E}}$; 1 the Sun passing the meridian; $\underline{\text{1}}\,\underline{\text{B}}$ its altitude; A B the altitude of the equator. $\underline{\text{1}}\,\underline{\text{A}}$ the declination of the Sun; ~~A E its right ascension~~ and $\underline{\text{1}}\,\underline{\text{E}}$ its Longitude; $\underline{\text{1}}\,\underline{\text{A}}$ is known as it is universally equal to the difference between the sun's altitude and that of the equator ~~A E is known~~ because ~~the two points A & E are given~~; therefore in the spherical triangle $\underline{\text{1}}\,\underline{\text{A}}\,\underline{\text{E}}$, in which the angle $\underline{\text{A}}$ is a right angle and the angle E equal to $23\frac{1}{2}$ degrees, the hypothenuse $\underline{\text{1}}\,\underline{\text{E}}$ is easily found. ~~Cos E : R :: T. A E : T. 1 E.~~ S. $\underline{\text{E}}$: R :: S. $\underline{\text{1}}\,\underline{\text{A}}$: S. $\underline{\text{1}}\,\underline{\text{E}}$. ~~S. bet. C. & 1 E : T. A E.~~ But $\underline{\text{1}}\,\underline{\text{E}}$ thus found is not always the sun's longitude but only its Distance from the nearest equinox. Let $\underline{\text{R}}\,\underline{\text{♑}}\,\underline{\text{♈}}$ be the the equator near



as a right line $\underline{\text{R}}\,\underline{\text{♑}}\,\underline{\text{♈}}$ the ecliptic; the first six signs above or to the north of the equator, the last six beneath or to the south; When the sun is in the northern signs and is nearest to the vernal equinox as at $\underline{\text{1}}$ the hypothenuse 1 ♈ found by the proportion will be the true longitude; at ♑ nearest to the Autumnal equinox $\underline{\text{♑}}$ will be the supplement of the longitude $\underline{\text{♑ 1 ♈}}$ to 180 degrees; when the sun has a South declination as at $\underline{\text{A}}$ and is nearest to $\underline{\text{♑}}$, $\underline{\text{♑}}$ added to 180 degrees is equal to the longitude $\underline{\text{♑ ♑ ♈}}$; at $\underline{\text{p}}$ near the vernal equinox $\underline{\text{R p}}$ is the complement of the longitude $\underline{\text{p ♑ ♈}}$ to 360 degrees.

Modern astronomers find the sun's distance from the nearest equinox without knowing the elevation of the equator or the sun's declination; by observing the sun at two different times of the year ~~when~~ as at E & G when it is in the same parallel to the horizon with a certain star, as E or at equal vertical distances from another, as L by this the horizontal distances E & G E (to the ~~star~~ to the vertical passing through it) are found the sum of which or B D is equal to twice B K the complement of ~~a~~ B or D S to 90 degrees, then Sin. S : R :: 1. D S : D. 1 S.

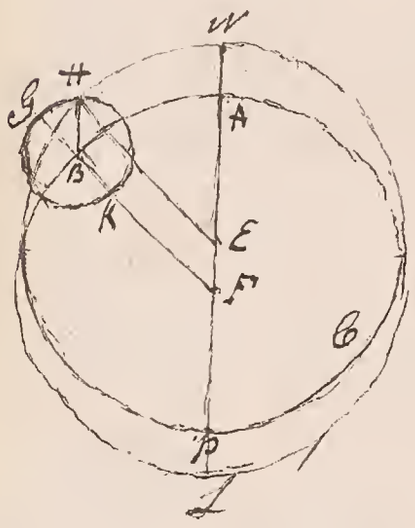
The day and hour of the equinox may be easily found by these rules as at the vernal equinox the longitude, declination and right ascension become equal to nothing as the sun is then in the first of aries; at the autumnal equinox the longitude and right ascension are equal each to 90 degrees and the declination vanishes.

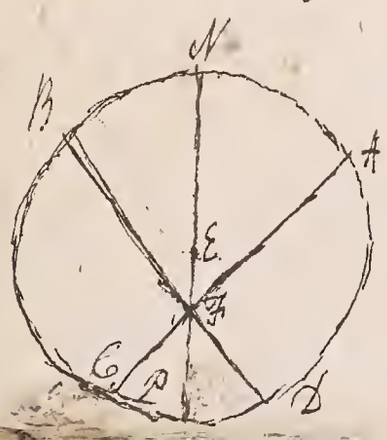
The length of the ~~year~~ is the interval between one vernal equinox and that of year following; the time is found more exactly by taking two ~~observations~~ corresponding equinoxes observed a thousand years distance and dividing the interval of time by a thousand; by this method the length of the year has been found equal to 365 day, 5 h. 48 ' 45 ".

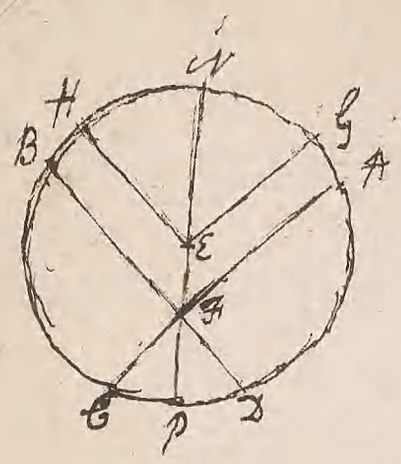
Knowing the sun's place in the ecliptic every day in the year, the irregularities of its motion may be found, by comparing the longitude observed with the longitude that the sun would have, were it to move round the earth uniformly in a circle, in the space of a year; the mean longitude is found by dividing 360 degrees or 1296000 seconds by 365 1/4 which gives 59 ' 8 " 3/10 the mean motion of the sun in 24 hours, which, multiplied by the number of days since the sun passed the equinox, gives the mean longitude.

The true longitude coincides with mean, in the beginning of the months of January and July; it is greater by two degrees or $1^{\circ}55'31''$ in the month of April and less by the same quantity in the beginning of October; this difference is called <u>the equation of the center</u> or <u>the equation of the orbit</u>.

The ancients formed two hypotheses to account for this irregularity; first, they supposed that the sun moved uniformly (because that was the most perfect of all motions) and in a circle (because that was the most perfect figure) round a center at a distance from the earth; the next hypothesis was that the sun moved uniformly in a circle (G H K) whose center (B) moved, uniformly, in the same time, in a contrary direction, ~~round~~ in a circle (H B L) round the earth as a center; first suppose the circle G B K replaced that the line (D B H) drawn through the earth (A) and its center (B) may pass through the sun, then let the center B move from A to B, the sun will have moved from G to H describing an arch G H equal to BA, join B B, G & B H, draw H E parallel to B A; then as the arcs G H & B A are equal the angles at B & B will be equal and B H & B A parallel, H A is therefore a parallelogram and H E equal to B A and A E to B H; but B A is a constant quantity therefore H E is constant, and therefore the path of the sun is a circle whose center is E, placed at a distance E A from the earth, equal to the radius of the circle G H K. These two hypotheses, which are the same as to the sun's path and apparent motion, accounted in some degree for its irregularities; for sun will appear from the earth to move ~~much~~ ~~quicker~~ with a great velocity at P than at L because L being more distant from E than P the space that the sun ~~move~~ describes every day will appear less at L than at P; now drawing the lines N C & B D through E







at right angles to one another and making the angles
H E A, G E B equal; the sun will describe G D sooner than
F A though the are apparently equal being both seen
under a right angle; Drawing E G & E H at right angles
H E C will be the quarter of the mean revolution and B H x G A will be
the suns inequalities.

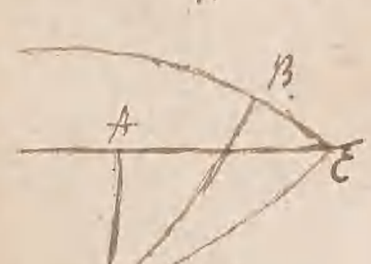
A the most distant point from the earth is called the Apogee;
P the nearest point, the Perigee; E F the excentricity; The suns distance
from its apogee, as F A, the Anomaly; When the earth is supposed to move
in an ellipse round A. earth the point A is called Aphelion
and the point P the Perihelion; the points D & A are called the
Apsides.

To find the Right ascension, declination, longitude and latitude of a fixed Star

To
Find the right ascension of a Star for a certain day; observe its passage over
the meridian and that of the Sun star given; convert the interval of time
into degrees and add it to or substract it from the suns right ascension;
the sum or the difference is the right ascension of the star. The right
ascension of one star being determined, that of all the others may be found
by comparing them with it, without having recourse to the Sun.

The Declination of a star is equal to the Difference between its
meridian altitude and that of the equator.

Knowing the declination and right ascension of a star,
to find it latitude and longitude. Let S A be the declination
of a star S, and A E its right ascension or its Distance from the
nearest equinox measured on the equator; let E B be the
ecliptic and S B the circle of the stars latitude; S B & B E are
the latitude and longitude required. Let S E be an arc of a
great circle passing through the equinox E and the star S; In the right=
angled spherical triangle S A E as the two sides S A & A E are known
hypothenuse S E and the angle S E A may be found. R: cos. S A :: cos. A E : cos. S E
& S. A E : R :: T. S A : T. S E A; by this the angle S E B is known, for it is equ
to S E A - B E A (or 23° ½) when S & B are on the same side of the ecliptic and to



$\odot E \Delta + B E \Delta$ when they are on contrary sides, then in the right-angled spher=
=ical triangle $\odot B E$ as the hypothenuse $\odot E$ & the angle $\odot E B$ are known,
the other sides may be found; $R : s. \odot E B :: s. \odot E : s. \odot B$ the latitude
of the star, required, and $R : \cos. \odot E B :: T. \odot E : T. B E$ the longitude of
the star required, or it distance from the nearest equinox.

The angle of position $B \odot \Delta$, contained between the circles of Latitude
and declination, is equal to the difference between the angles $B \odot E$ & $\Delta \odot E$.
when $\odot$ & B are on the same side of the equator and to their sum when
they are

The longitude of of the stars increases constantly, that is, the equinox
or the point of intersection of the ecliptic and equator moves in a
direction contrary to that of the sun's annual revolution; this motion
is for that reason called the precession of the equinox; it is calculated
to be equal to 1° „ $23'$ „ $10''$ every century, this quantity is supposed to be
not quite uniform.

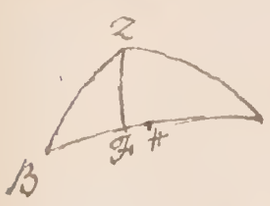
As the equinox moves in a direction contrary to that of the sun,
the sun will meet the equator before it reaches that point in the
heavens where the equinox happened the year before; therefore
the tropical or solar year or the distance of time between two
equinoxes must be less than the sydereal year
or the time in which the sun performs a complete revolution
of the heavens: the first is calculated to be 365 days „ 5^{h} „ $48'$ „ $45\frac{1}{2}''$
and the other 365 days „ 6^{h} „ $9'$ „ $11''$.

The observation of equal altitudes.

It has been already proved that ~~the sun~~ a star, in its diurnal ~~motion~~ motion, at equal altitudes before and after passing the meridian, is at equal distances from it; ~~the same thing is true of the stars;~~ the diurnal motion being uniform, ~~these distances~~ equal distances will be described in equal times. A method, therefore of finding the exact time of a ~~heavenly body's~~ star's passing over the meridian is to observe its altitude at a certain moment before it crosses the meridian, and the moment in which it arrives at the same altitude after having crossed it; the moment that bisects that interval of time is that of the ~~heavenly body's~~ star's crossing the meridian. the middle time is an arithmetic mean between the other two observed. In general several corresponding altitudes, are observed, and a mean taken of their results

The time of the Sun's passing the meridian is likewise found by the method of equal altitudes, though a small correction is required for the ~~sun's~~ change of Declination ~~in the ecliptic~~. Let A E Z P B be the meridian, Z the zenith, P the pole of the earth; Let S & B be two points of equal altitude in the Sun's Diurnal arc S E B; ~~join~~ let P S & P B be two secondaries to the equator passing through S & B & Z S, Z B two verticals passing through the same points; let E be the point in which the Sun passes the meridian, E P S & E P B are horary angles and measure the times that the Sun takes up ~~employs~~ to describe the arcs S E & E B; Were the Sun to move in the equator or in one of its parallels, (the secondaries P S & P B, that measure its declination before and after its passing the meridian, would be equal, and in the spherical triangles Z P B & Z P S, Z P being common & Z B, Z S (the complements of the equal altitudes) equal, and likewise P B & P S equal,) the angles S P E & B P E would be equal, the Sun ~~would~~ described S E & B E in equal times, and the moment that bisects the time of the Sun's moving from S to B would be exactly that of its crossing the meridian; But when the Sun's declination changes P B & P S become different and their difference equal to the change of Declination, and in the spherical triangles Z P B & Z P S, Z P being common & Z B, Z S equal, when P B is less than P S the angle Z P B is greater than Z P S and less when it is greater, therefore when the Sun's declination increases E P B becomes greater than E P S and less when it diminishes

that is, in the first case, the time of the sun's ~~pa~~ moving from E to B is greater and in the second less than that of its moving from I to B; therefore the moment that bisects the time of the sun's moving from I to B will in neither case be that of its passing the meridian but will fall, in the first case between E & B and in the second between E & I; as half the sum of two unequal quantities is equal to the least added to their difference or to the greatest diminished by it, half the difference of the angles BPE & EPI converted into time and subtracted, in the first case, from the hour found by the moment of bisection and added to it, in the second, will give the corrected time of the Suns passing the **meridion.** In the triangle ZBP, the three sides being known, the



angle ZPB may be found; Let ZF be part of a great circle passing through Z at right angles to BP ~~and H~~ bisected in H.

Tan. $\frac{1}{2}$ HB (or PH) : R :: $\frac{BZ+ZP}{2}$: Cot. $\frac{BZ-ZP}{2}$: T. HH; HH being known PF is known and in the right angled spherical triangle PFZ, R : PF :: T. PZ : Cos. P ; the angle ZPB in the spherical triangle ZPB may be found by a similar proportion. Therefore the angles ZPI & ZPB being ~~known~~ found, their difference is known. Fig. 1.

~~The measure of time.~~

~~A day, the space of time in which the sun performs one diurnal revolution round the earth, is divided in to 24 equal parts called hours. As the sun is constantly changing its right ascension in a direction contrary to that of the diurnal motion, the sun's motion will be slower than that of a fixt star, ~~Primum mobile~~; ~~therefore~~ when the sun has passed the meridian at the same time with a fixt star and the same star, after a complete revolution, is come again to the meridian, the sun will not yet have reached it. ~~will be by [illegible] it, this difference is called the accelleration diurnal accelleration of a star.~~~~

The measure of Time.

A Day, or the space of time in which the Sun performs one diurnal revolution round the earth, is divided into 24 equal parts called hours, each hour is divided into 60 minutes and each minute into 60 seconds.

The Sun performs its annual revolution or describes the space of 360 degrees in 365 days, 6 hours, 9 minutes & 11 seconds; this motion, supposed uniform will describe spaces proportionable to the times in which they are described, therefore $365^{Dayes}_{,,}\ 6^{h}_{,,}\ 9^{m}_{,,}\ 11^{s} : 360^{\circ} :: 24^{hours} : \underline{59',8,''192}$ the portion of the Sun's annual orbit ~~supposing~~ described in 24 hours, supposing it to move uniformly, or its mean change of right ascension in a day. By this motion of the Sun in a direction contrary to that of the Primum mobile (or the diurnal revolution of the heavens) the velocity of its diurnal motion will be less than that of the fixt stars; if, therefore, the Sun pass the meridian at the same time with a fixt star and after a diurnal revolution come to it again the star will have passed it; the star's distance from the meridian or the difference between its right ascension and that of the Sun will be equal to 59',8,''2 (equal within the $\frac{1}{100}$ part of a second to 59',8,''192) the Sun's change of right ascension, because the right ascension of the star has not changed.

The Sun describes 360 degrees and the star 360°,59',8,''2 in 24 hours, therefore in one hour, the sun describes 15° and the star 15°,2',27,''8 and the velocity of the sun and star are to one another as 15° to 15°,2',27,''8.

The motion of the primum mobile is uniform therefore 360°,59',8,''2 : 24^{h} :: 360° : 23^{h},56^{m},4,^{sec}09 time in which a star performs a complete diurnal revolution.

$23^h\ 56'\ 4''09$ subtracted from 24^h hours gives $3^m\ 56^{sec}$ the time in which the star describes $59'\ 8,2$ or the time between the passage of the sun and star over the meridian; this is called the Diurnal acceleration of a star.

Clocks in common use as well as the greatest part of astronomical clocks are regulated on the mean motion of the sun, that is, were the sun to move uniformly the clock would mark 12 every time the sun passed the meridian. Some astronomers have regulated their clocks by the stars so that every time the first of Aries passed the meridian the clock marked twelve; and made their 24 hours the time of a revolution of the primum mobile; therefore for every hour that the clock marks the stars will describe 15 degrees and the hour marked by the clock is always equal to the right ascension of the culminating point (or the point of the equator passing the meridian) reduced to sidereal time.

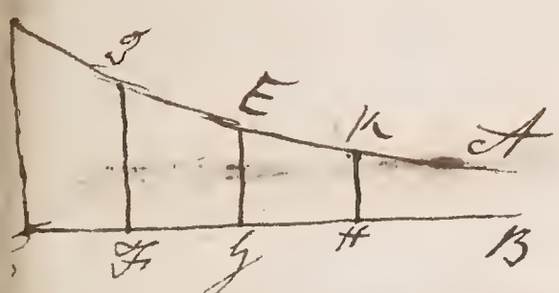
If a clock's motion be uniform its velocity is of no consequence; for the time it counts during a known interval may be easily found and, as its motion is uniform, this time will be to the time it counts during another interval as the first interval to the second: if it gains or loses a certain quantity during a diurnal revolution of the Sun, the time it has counted during that interval is to 24 hours as the time it has counted during any other interval to that interval reduced to solar time.

The Equation of Time.

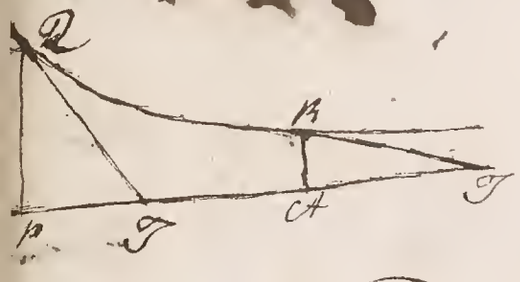
The Sun's diurnal revolution, as determined by the observation of equal altitudes, is not a proper measure of time, because its change of right ascension is not uniform, and consequently the time in which it performs a diurnal revolution varies; This variation depends upon two causes, the sun's irregular motion in the ecliptic and the obliquity of the ecliptic to the equator on which the right ascension is measured; the first is equal to difference between the sun's true and apparent longitude, the second between its true longitude and true right-ascension and the two combined, or the equation of time, to the difference between the true longitude and the ~~true~~ mean right-ascension; as the mean right ascension is found by supposing a sun that makes a complete revolution round the equator in a year and that marks the mean time the right ascension of this sun will be always equal to the suns mean longitude.

The Sun's change of right-ascension is irregular, its diurnal revolution as determined by the method of equal altitudes is therefore an improper measure of time; mean time is determined by the instant at which the sun would pass the meridian were its motion uniform, the apparent time by that at which it actually does pass it; the interval of time between these two instants is called the Equation of time.

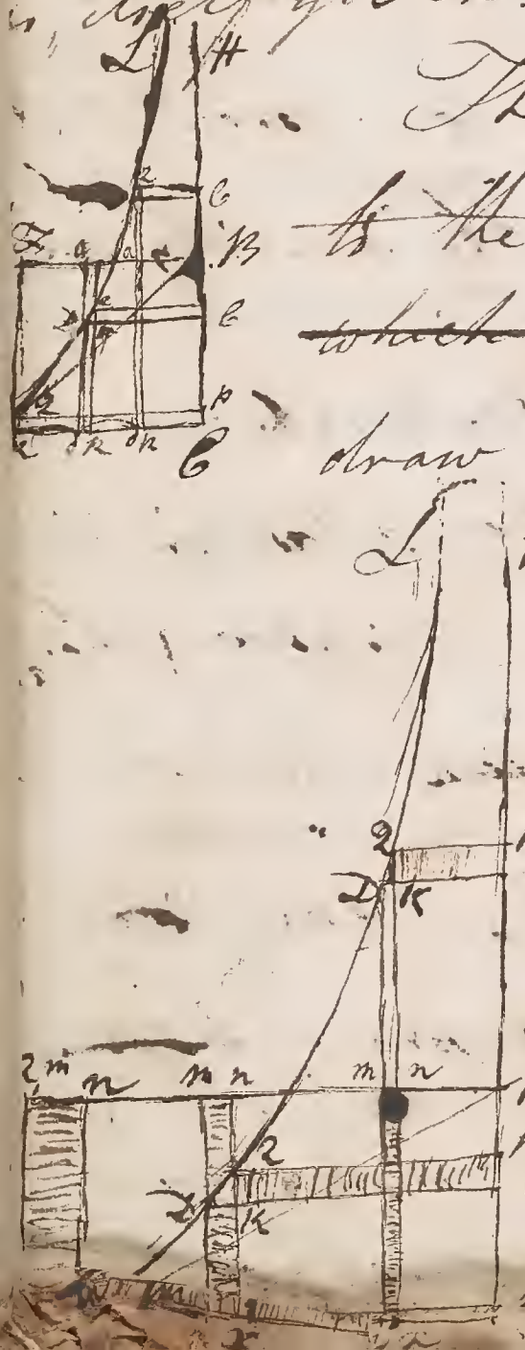
The Logarithmick curve



Let AEg be the logarithmick curve, CB its asymptote produced
does not meet it till produced to an infinite length, let Ff, yE & HK
ordinates to the curve, the segments of the asymptote intercepted between
so of the ordinates are the logarithms or measures of the ratios they have
to one another, therefore if abscisse CF, Cy, CH &c are in geometrick
rithmetical progression, the corresponding ordinates Ff, yE, HK &c will
e in geometrical progression.



In a given logarithmick curve, the subtangent of (or
that part of the asimtote intercepted between a tangent
and an ordinate, drawn through its point of contact) is a given
quantity. Draw QP an ordinate very near to BC, and from Q draw QR at
right angles to BC; then, as CP is given, the ratio of Ct to (QP or) Rt
will be given and of Ct to (Ct - Rt or) BR is given; the triangles
BRQ & BCt are similar (as BQ which is very small may be considered
as a part of the tangent at B) therefore BR : Ct :: RQ : Cτ therefore
Cτ has a given ratio to a given quantity (QR or) Ct P and consequently
itself given.

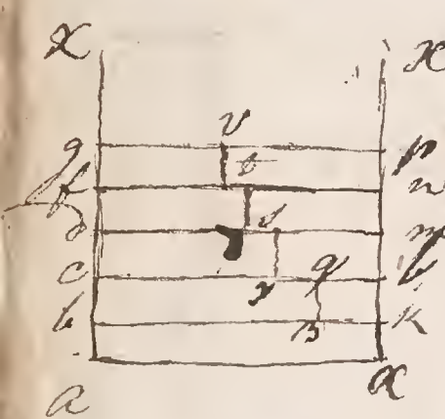


The indefinite area DCHL cut off by an ordinate DC is equal
to the rectangle under the subtangent BC and the ordinate DC
which it is to be
Take any point D in the curve, through it
draw an ordinate DC and a line mD parallel to BC; draw
another ordinate dp very near to DC and at the given
distance Cp; then, as it was proved before, DC : Dk ::
BC : dK and DC × dK = BC × DK that is, the rectang
Dx is equal to the rectang mx, from what
point in the curve the lines DC & mD, dy are drawn; t
the space DCHL is composed of the rectangles
Dp, and the space dCB of the rectangles mx which
equal
respectively
and DCB are

As the area D C H L is equal to a D C B, and that a C is as its base, the ordinate, (its altitude C B being given) D C H D is as the ord by which it is bounded.

~~The density of the air is as the force by which it is pressed.~~

The densities of the air at different distances from the earth's surface are as the ordinates, of a logarithmick curve, drawn through at their respective altitudes.



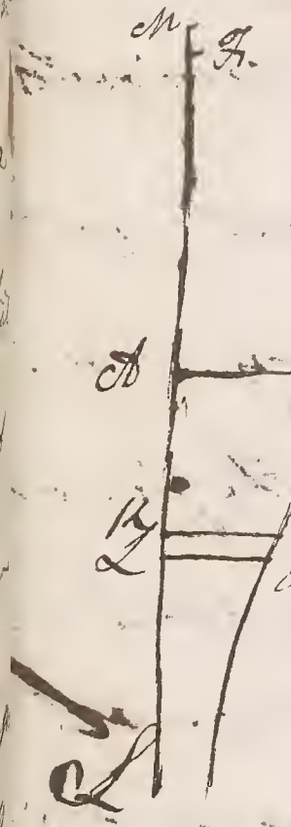
Let x A x x represent a vessel reaching from the surface of the atmosphere x x and suppose the side a x divided into any equal parts as inches a b, b c, c d, d f &c and let lines a k, c b, d m &c be drawn through each of the points b, c, d &c. It is evident (as the density of the air is as the force by which it is pressed) that the air between c b & b k will be less dense than that between b k & a, as b k has a greater quantity of air to support than a x, for the same reason, the air between d m & c b is more rare, than that between c b & b k and every inch to the top of the atmosphere is more rare than those below it. Now suppose the density of the air to be the same in every part of the same inch, but more rare in every inch than in that below it; and let the air in each of the inches be so prest that its density be the same with that of the air in a k; the density of each inch is as the quantity of air in the spaces a k, b g, c r &c (the magnitudes being given) or as the same quantities of air in the compressed state in the spaces a k, b g, c r &c which are as those spaces (as the density is given) therefore the density of the air in every inch is as the space which it occupies when made of equal density with a k. Again as the density of the air is as the force by which it is prest the density in each inch will be as the weight of the whole atmosphere above it or as the sum of all the rectangles into which it is pressed: therefore

in every inch, the sum of the rectangles above it, is as the rectangle into which the air contained in it, is presses, to be of equal density with ak.

And if any number of attitudes be taken in ~~geometrical prog~~ arithmetical progression the sums ~~of the rectangles~~ above each attitude will be ~~in~~ in

a:a-b::b:b-c ab-ac=ab-bc

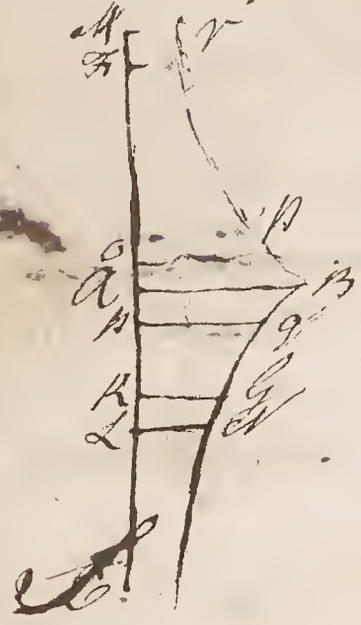
geometrical progression, as they are to one another as their differences, which will likewise be in geometrical progression, and as these rectangles have a given attitude, they are, ~~therefore~~, as their bases the lines whose extremities will be a logarithmick curve to which they are ordinates in geometrick progression, whilst the parts of the abscissa x are in arithmetic progression. In the same manner it may proved, by taking the attitudes indefinitely small, which before supposed to be inches, that the density at any attitude from the surface of the earth is as the ordinate to the logarithmick curve.



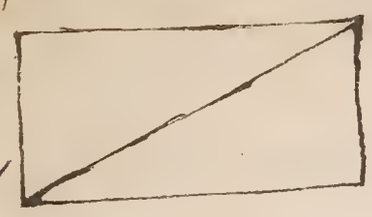
Allowing for the difference of gravity.

Let L be the center of the earth and AB part of the surface let LA be produced to the extremity of the atmosphere and let F be a point in that line, at which the density of the air is required; take LK a third proportional to LF & LA; through K draw KG representing the air's density at F; let M be another point at an indefinitely small height above F, and take LM a third proportional to LM, LA and through L draw LN representing the air's density at M. The curve passing through B, G & N is the same logarithmick curve with the last, but in an inverted position. For since LA is a mean, as well between LM & LL as between LF & LK, by compounding and dividing the ratios, and by supposing LL & LK equal to one another, (their differences being inconsiderable,) LK is to MF (as LK to LF) or as LA² to LF² and LKL = $\frac{LA^2 \times MF}{LF^2}$ as LA² is a given quantity KL : $\frac{MF}{LF^2}$; but MF is as the quantity of air in the space MF

in the space FeM and fEg, is inversely as the force of gravity, therefore

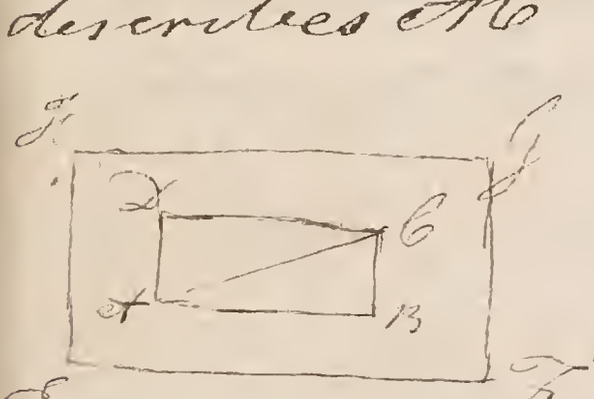


the rectangle under KL and Kg, the density at F, or the area KGNL will be as the gravitation, density & quanty of the air ~~at small space MF times~~ in the small space MF, that is, as its pressure upon the air ~~at F~~ beneath it; ~~consequently the pressure upon any point F is as the sum~~ of the similar areas ~~below the corresponding point~~ and the sum of all the similar areas below KG will be as the sum of the prepsures upon F, or as the density of the air at that point, that is as KG by which the density is represented; And KGNL, the difference of two of those sums one of which begins at K, and the other at L, is as the difference of the densities of the air at the points F & M, or of KG & LN. Now let KL be given, then Kg will be as the area KGNL or as KG-LN, Therefore Kg is as LN, that is the ratio of KG to LN is given and consequently is measured by the given line KL; but this is a property of the logarithmic curve, therefore the line passing through the points K, G is a logarithmick curve, and the same, in every respect with the former but in an inverted position; for, taking AO a very small height above the earth's surface, it is evident that the force of gravity is the same at B or at A and the density the same as if the difference of gravity was not considered; take AB equal to AO; then as the density at the earths surface was represented by the equal ordinates AB, the ordinates OP & BQ which represent the density at the point B will be equal to Bg at O, will be equally to one and the; therefore the two logarithmick curves Bpp & Bg GN have the same curvature and are similarly situated on the opposite sides of their common ordinate AB; AB represents the air's density at the surface of the earth and Kg at the point F it is evident that AK will be the measure of the ratio of the density at the surface to that at F, therefore if the density at F is required AK must be found by taking AK a third proportional to AF, AE; for by division AK will be to AF as EA to AF.

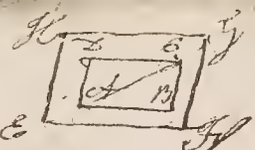


1. Let the body A have a motion in the direction AB, represented by the line AB; and let another motion be communicated to it in the direction AD, represented by the line AD; Complete the parallelogram ABCD, and A will proceed in its diagonal AC, & will describe AC in the same time that it would have described AB, AD with their respective velocities.

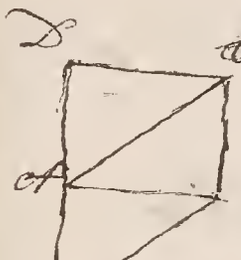
Because AD is parallel to BC, a motion communicated to A in the direction AD will neither promote nor retard its approach towards BC; therefore it will arrive at BC in the same time as if it had been acted upon only by the motion in the direction AB. In the same manner it may be proved that as AB is parallel to DC, A will arrive at DC in the same time as if had been acted upon only by the motion in the direction AD. Therefore A arrives at BC & DC in the same time that it would have described AD, AB with their respective velocities; but A can only arrive at BC, DC (in the same time) in their intersection. Therefore AC is the diagonal of the parallelogram, and A describes AC in the same time & 2 CD



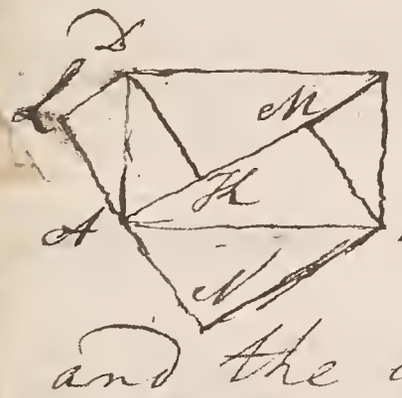
2. Suppose the Space EFGH to be carried in the direction AB with an uniform motion represented by the line AB; and let a body A, in that space, by acted upon by a force in the direction AD and measured by the line AD. Then, to those who are in the Space EFGH, A will appear to move in the right line AD, but it real motion will be along the diagonal



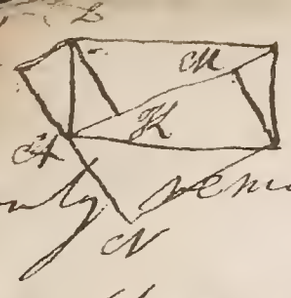
diagonal AG, and it will describe AG in the same time that A or any other point in the Space EFGH is carried oder a line equal to AB or that A would have described the line AD, if the space had remained fixt. For AD, by the motion of the space, will by carried into the situation BG and the point D to G there A really moves in the diagonal AG.



2 The converse of this is that the motion in the diagonal AG may be resolved into the motions in the sides of the parallelogram AB, AD. Take AK equal to AD and in an opposite direction, complete the parallelogram & AB will be its diagonal then GK compounded with AK equal and opposite to AD will produce AB; that is if the motion in AD one side of the parallelogram be subducted from the motion in the Diagonal AG there will remain AB the motion in other side.



Let the forces in the two sides AB, AD be each of them resolved into two other forces one in the direction AG and the other perpendicular to it; that is let AB be resolved into the forces AM, AN, the one in the direction AG, and the other perpendicular to it and AD into AK, AL; then as the triangles DAK, BAG are equal and similar the perpendiculars DK, BM are that is the lines LA, AN are equal to one another and as they are likewise opposite the will destroy each other's effect, Therefore if AK be compounded with, AM it will produce AK + AM, but AK is equal to AG because AK = AG, or AG, because AK = AG therefore A will move in the diagonal AG and as AK is equal to MG, the compound motion of AK, AM will be expressed by AG. From this it appears that absolute motion is lost in the composition of motion, because the parts, LA, AN, of the compounded motions of LA, AB being equal and opposite destroy each others effect and the other parts AK, AM only



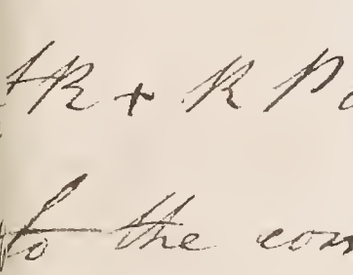
only remain in the direction of the compounded motion of A C. But

on the contrary, absolute motion is gained in the resolution of motion,

because ~~of~~ A B + A D or B C is greater than A C. But if compound motion

is estimated in a given direction it will neither loose by composition

nor gain by resolution but will remain invariably the same.



Let the forces D A, A B be estimated in the given direction

A P to which let perpendiculars be drawn from the

points C, B & D; then as A C may be resolved into the forces C P, B A

and that C P, being perpendicular to A P, has no effect in that

direction the force A C in the given direction A P is equal to A P;

in the same manner the force A B is equal to A R and A D to A Q

in the given direction A P. Then as R P is to B S or (B C or) D A to

A S or A Q to P S therefore A Q is equal to R P and A Q + A R is equal to

A R + R P or) A P; that is the sum of the forces A B, A D, reduced to any given direction

is equal to the compounded force A C reduced to the same direction. From

this it is obvious that if any number of forces be compounded together

or any force be resolved into any number of forces their sum will

remain invariably the same till some foreign influence acts upon

them acted upon by some foreign influence.

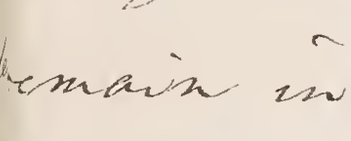
If three forces act upon a body so as to retain it in equilibrio

they will be to one another as the sides of a triangle drawn parallel

to their respective directions, as the sides of a triangle no sides

of which perpendicular to those of the first to and as the sines of the

angles through which they pass.



1st The forces B A, D A & F A which retain the body A

in equilibrio are to one another as the sides of a triangle to which

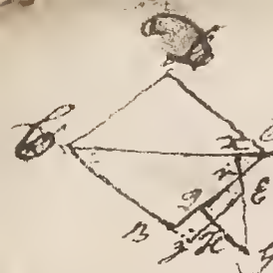
~~D A~~ they are parallel. Through B draw B C ∥ D A & through D draw

D C ∥ A B; then as the force A F is equal to that compounded of A B, A D.

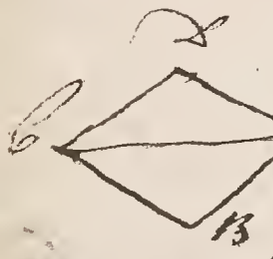
that is to A C, and that C B is equal and parallel to D A, A F: D A:

:: A C: C B and D A: A B :: C B: A B.

2 dly



2dly The forces AB, AC, AF will be to one another as the sides EG, GH, HE of a triangle perpendicular to them, because the angles at x, y, z are equal to two right angles, the complements E & C will be together equal to two right angles and will be equal to XEH & XEG together, from those equals take away the common XEH and there will remain HEG equal to AEB, because the angles at X & Z are right angles and the verticals at E equal the angles at A & G are likewise equal and therefore the triangles HEG, ABC are similar and have their sides as the forces, to which they are parallel.



3dly These forces are to one another as the sines of the angles through which they pass. DA (or CB) is as the sine of the angle (CAB or) BAF through which it passes & BA is as the sine of the angle DAF through which it passes therefore DA will be to AB as the sine of the angles BAF to the sine of DAF; in the same manner, AF will be to DA as the sine of the angle DAB through which it passes to the sines of the angles through which the bs.

Cor: Any three forces which retain a body in equilibrio may be represented by the three sides of a triangle; of which any number may be drawn upon the same base, if their vertical angles are equal the locus of all the vertical points will be in the segment of a circle and their common base a line inscribed in that circle, if the sum of the sides is equal in all of them, then the vertical points will be in an ellipse and their base that segment of the major axis between the foci, if the areas of all the triangles are equal the points will be in a straight line.

2d Sheet

The proportions of Force, Motion, Space, & Velocity, Time and Quantity of matter are found as follows.

$$F : m : qv : \frac{qs}{t}$$

$$m : F : qv : \frac{qs}{t}$$

$$S : tv : \frac{tm}{q} : \frac{ft}{q}$$

$$V : \frac{s}{t} : \frac{m}{q} : \frac{f}{q}$$

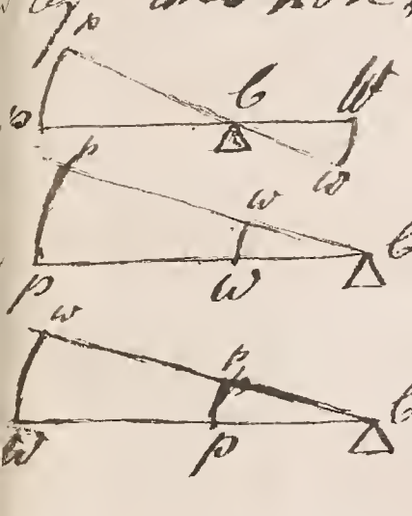
$$T : \frac{s}{v} : \frac{m}{f} : \frac{sq}{m} : \frac{sq}{f}$$

$$Q : \frac{f}{v} : \frac{m}{v} : \frac{st}{s} : \frac{mt}{s}$$

If one body AB is struck by another C in an oblique direction, the force impressed upon AB is to be estimated only by the perpendicular of C represented by Cc. Let the force of C be resolved into the two others CD & CE. Let the force CE, one of which CD is perpendicular to the other, be supposed parallel to AB & suppose C to move in that direction, then CE can have no effect, therefore C will move in the direction CD & strike O as if the force AB had not been impressed, therefore AB is acted upon only by the force CD.

Of the Mechanical Powers

1st The power and the weight acting at right angles will be in equilibrio on a straight lever when they are inversely as their respective distances from the centre of motion; that is when P : W :: WC : PC.



Let the lever be moved round the Centre so that PW may describe the arches Pp & Ww; then as these arches are described in a given time, the velocities with which P & W move may be represented by them, and as the sectors CPp, CWw are similar, they (the arches) will be as PC : WC but then will be an Equilibrium when the Quantity of matter is inversely as the Velocity Therefore P : W :: Ww : Pp; then mP : mW :: P × Pp : W × Ww but P × Pp = W × Ww ∴ mP = mW the quantity of motion in P equal to the quantity of motion in W therefore the power and the weight

&c Q E D

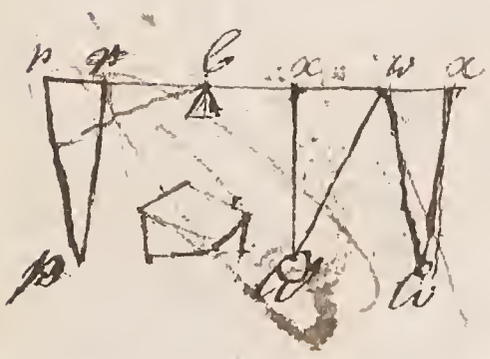
Any number

any number of forces will be in equi: on a lever when the sum of the products, which arise by multiplying each force by its respective distance, on one side of the lever is equal to the sum of the products on the other side of the centre of motion

As it was proved before, if $P \times p = W \times w$ (that is if $P : W :: w : p$) P would be in equi: with W; in the same manner if the sum of any number of rectangles or forces on one side be equal to that on the other, the sum of the weights and powers must be in equilibrio.

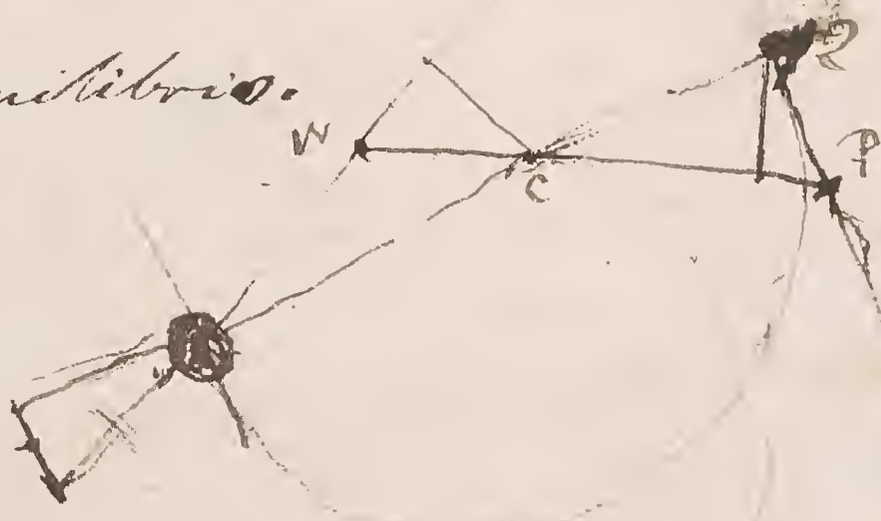
If P & W, acting obliquely are inversely as the distances from the centre of motion and perpendiculars let fall from them upon a straight lever; they will be in equilibrio.

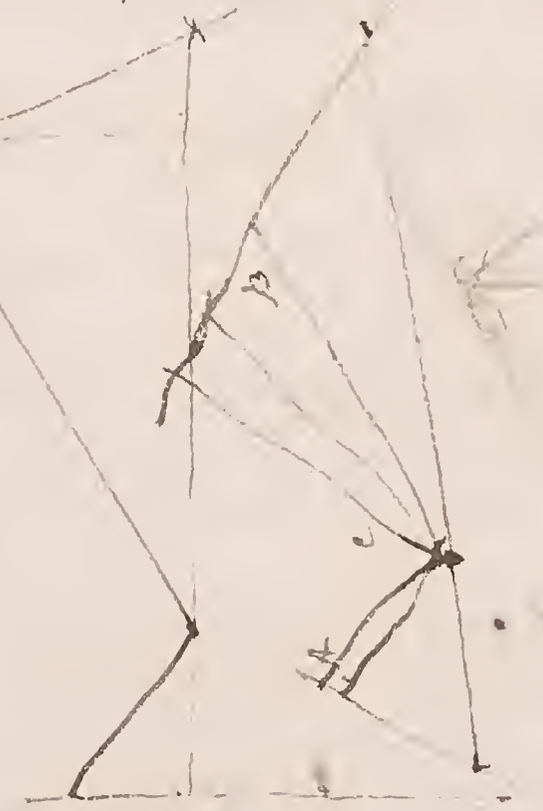
perpendiculars of their lines of D



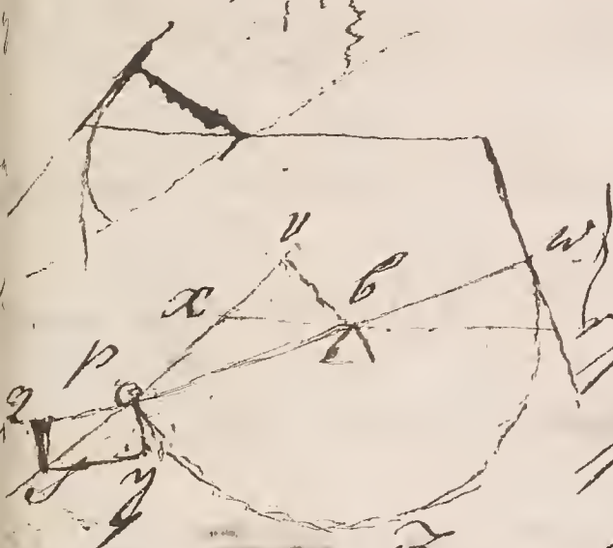
Let fall the perpendiculars Pp Ww &c. then if Pp to Ww inversely as C p to C x. P, W will be in equilibrio

let the force in the direction W w be resolved into W x, x w which last can have no effect as the point C remains fixt therefore W acts only in the direction W x; in the same manner it may be proved that P acts only in the direction P p, and these lines are per and they will have the same power at x or p as at any other points in those lines and by the hipothesis the are to one another inversely as their distances from the centre, therefore by the first case they will be in equilibrio.





§ If two powers be kept in equilibrio and act obliquely on the ends of a straight lever in oblique directions they will be to one another inversely as the perpendiculars from the centre to their lines of direction.



Let P & W act upon the straight lever in the oblique directions Px, Wr; and let bv, bw be perpendiculars to those directions from the centre; P will be to W as Wb to Vb. Let a circle be described about the centre b with the interval bW which will therefore touch Wr in W, and let it meet the line of direction Px in any point, then as P acts in that direction it may be placed at that point, let P1 represent the whole force in P, join bP and from 1 draw 12 perpendicular to it produced. Then the force P1 will be equal to the forces P2, 21 for P2 then as Wr is a tangent the whole force of W will be employed in describing the circle W2P and as the point b is fixed P will act in the circle only as 2b or P1, then as P and W are in equilibrio their forces in the circle must be equal therefore 21 is equal to the force in W, then as the triangles P12 & PbV are similar P will be to 21 that is P as the tangent to W as (bP or) bW to bV.

If two powers be kept in equilibrio on a bended lever they will be to one another inversely as the perpendiculars to their lines of direction from the centre

Let P & W be in equilibrio upon the bended lever 2b1, in the directions 2P, 1W, let bv, bx be perpendicular to those directions, bv to bx as W to x; draw through the point b any right line meeting 2P & 1W in any two points, then P & W may

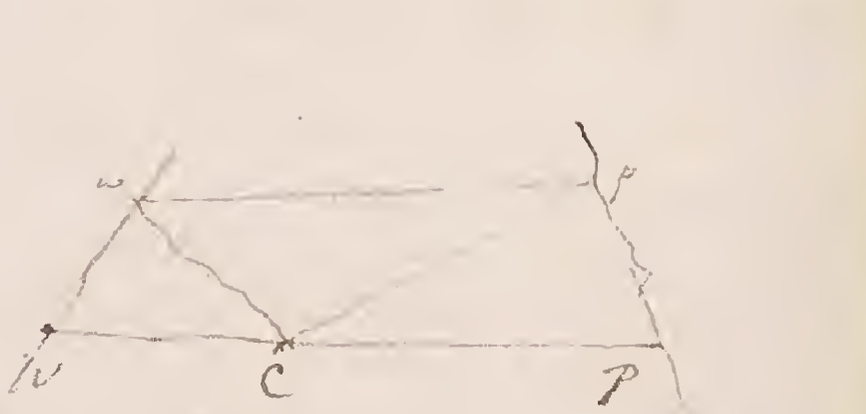
P & W may be placed at those points, they will be in equilibrio on the straight lever P C W and (by the preceding) ~~P.C~~ P : W :: C v : C x.

Let the two powers V P, V W act at right angles ~~to~~ to the bended lever W C P and ~~let their~~ set their lines of direction meet in V. Then ~~the~~ the point C will be acted upon in the

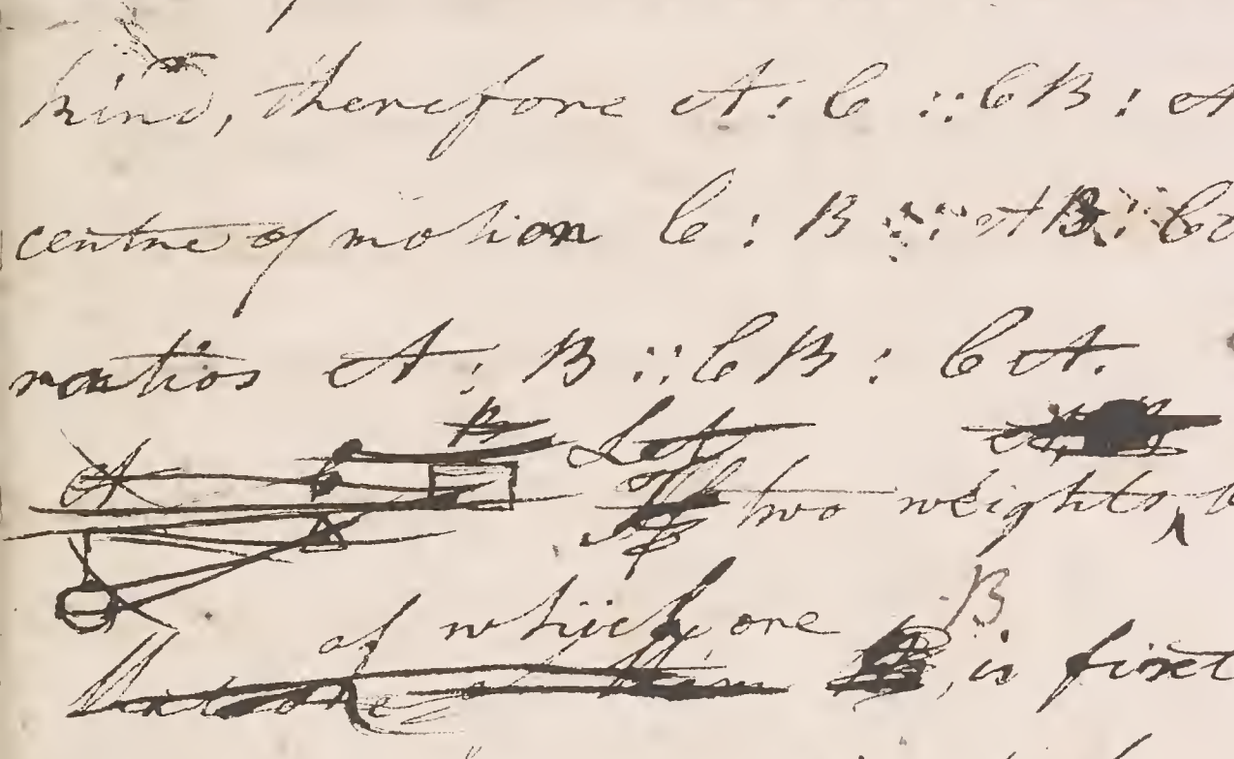
* the P, W, & the force on C will be directed

~~power~~ in the direction V C, ~~and to be~~ keep position equilibrio, join W P and draw C x parallel to ~~C V~~, W V; by the triangle C V X will be similar to W C P formed by the arms of the lever and the line joining its extremities. For the angles at W & P are two right angles, those at C & V are equal to the same, therefore a circle may be described about the quadrilateral W V P C; and as the angle W P C is in the same segment with W V C it is equal to it or to its alternate V C x, and C W P, C V X are equal being in the same segment therefore the triangles C V X & W C P are similar. Therefore the Power Weight x Acting upon C are as each to the distance of the objects.

And the power weight & posture of C in a straight line W C P when they act in the same direction as the forces acting C up & up.



If two powers at the extremities of a lever support a weight suspended upon it between them, they will be to one another inversely as their distances from the centre



Let the two powers A & B support the weight C, so A : B :: BC : CA. Suppose a prop to be placed at B instead of the power, then AB will be a lever of the second kind, therefore CA : C :: CB : AB and again, making A the centre of motion C : B :: AB : CA then by compounding the ratios A : B :: CB : CA.

Let two weights be in equilibrio on a horizontal lever of which one is fixed to the extremity; then if the centre of gravity in B be above the lever and that (the lever) be moved round upon its centre, as B falls it will require a greater weight than A to keep it in equilibrio and when it rises will be balanced by a less and the contrary will happen if the centre of gravity be beneath the line.

Let C be the centre of gravity right angles to the horizon; then if the lever AB be considered as parallel to the horizon, C will will act upon it at right angles and cut it in B then if the lever is turned about, C will act obliquely by C B as B is above or beneath that line, therefore the force on the side of the lever on which B is, will be less or greater as it is above or beneath the lever. In the same manner it may be proved that the contrary will happen if the the centre of gravity be beneath the levier

If a man ~~standing~~ in equilibria one of the scales of a Balance &
of a balance in equilibria either on the side of the centre of motion
in which he is, or on the opposite, the scale in which he stands
will preponderate.

Let D represent the force impressed upon the lever at the point D, then the force in D will be D x D C, and as action and reaction are equa
C will be pressed downwards with a force equal to D
and as C acts at C, D x C B will be the force with
which the lever is pressed down at the point C,
then as C B is greater than D b, C will preponderat
and if the force is applyed to the other side of the
lever, it is evident that both the force and reactio
act so as to make C preponderate because C will
acts at C But if he pulls down the beam at D the weight
C will preponderate.

S
X

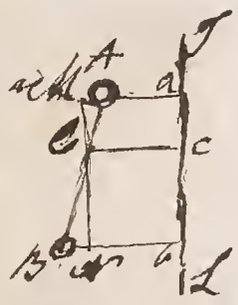
Of the centre of Gravity.

The centre of gravity is that point in a body or in a system of bodies (supposing them to be joined by an inflexible rod void of gravity) which being supported the whole will be in equilibrio.

Rule. In estimating the gravity of any body the whole is supposed to be collected in its centre of gravity, and to act in a direction perpendicular to the surface of the earth and as that may be considered as a plane, the directions of the gravities of different bodies may be called parallel, without any sensible error.

If the centres of gravity of two bodies be joined by an inflexible rod void of gravity, and a point be taken in that rod so that the two bodies be inversely as their distances from it, that point is the centre of gravity of the two bodies. For suppose a fulcrum to be placed at the point then the rod will become a lever of the first kind and by the hypothesis the two weights at its extremities will be in equilibrio; If there be another body in the system, let it be joined to the Centre of Gravity with centre of the other two, as before, by an inflexible rod, then, as the two first bodies may be considered as acting at their centre, if their sum be to the third body inversely as their distances from a point taken in the last rod; that point will be the centre of the whole system; the same thing may be proved of any number of bodies.

The sum of the products of any number of bodies into their respective perpendicular distances from a line or plane given in position, is equal to the product of the sum of the bodies multiplied into the perpendicular distance of their common centre of gravity from that line or plane, if the bodies are all on the same side of it, but when some of them are on the opposite side their products are with their respective distances are to be considered as negative

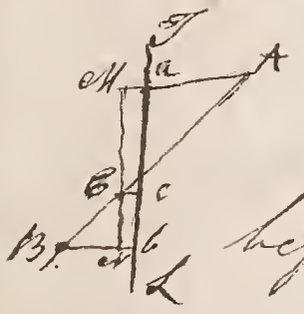


Let A & B be two bodies whose centre of gravity is C and whose perpendicular distances from the line or plane IL are Aa & Bb and Cc that of the centre C; then $A \times Aa + B \times Bb = A+B \times Cc$

Through C draw MN parallel to ab and meeting Aa & Bb in M & n; then as C is the centre of gravity of A & B, A : B :: (BC : CA

:: Bn : AM therefore $A \times AM = B \times Bn$, and, as $A \times Aa + B \times Bb =$

$= A \times Cc - A \times AM + B \times Cc + B \times Bn$, $A \times Aa + B \times Bb = A \times Cc + B \times Cc = A+B \times Cc$



Now suppose A to be on the opposite side of IL to C; then as before $B \times Bn = A \times AM$ and $B \times Bb - A \times Aa = B \times Bn + Bb$

$= A \times AM - A \times Cc = B \times Cc - A \times Cc = B-A \times Cc$. This the same is true of any number of bodies.

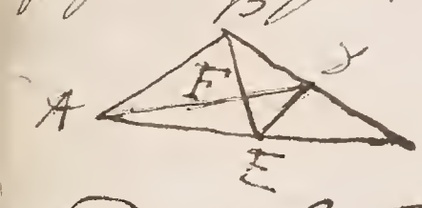
If a line be drawn from the centre of gravity of a body perpendicular to the horizon, if it falls within the base on which the body rests the body will stand, but if it falls without, it will fall down.

The centre of gravity will always seek the lowest point possible, as the weight of the body is supposed all to act in that point.

Methods of finding the centre of gravity in single bodies Direction or in a system of bodies.

First in planes of uniform texture; the centre of a circle is its centre of gravity, for every diamiter cuts it in to two equal parts.

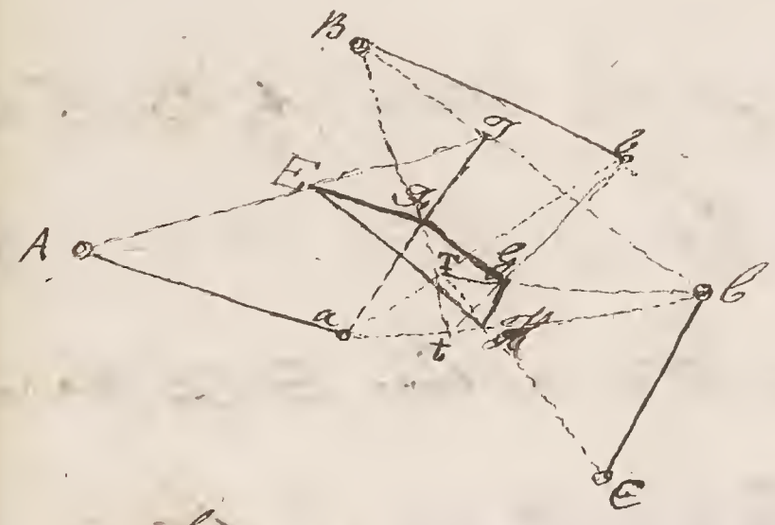
In a triangle, draw lines from two of the angles bisecting their opposite sides, the intersection of those lines is the centre of gravity of the triangle.



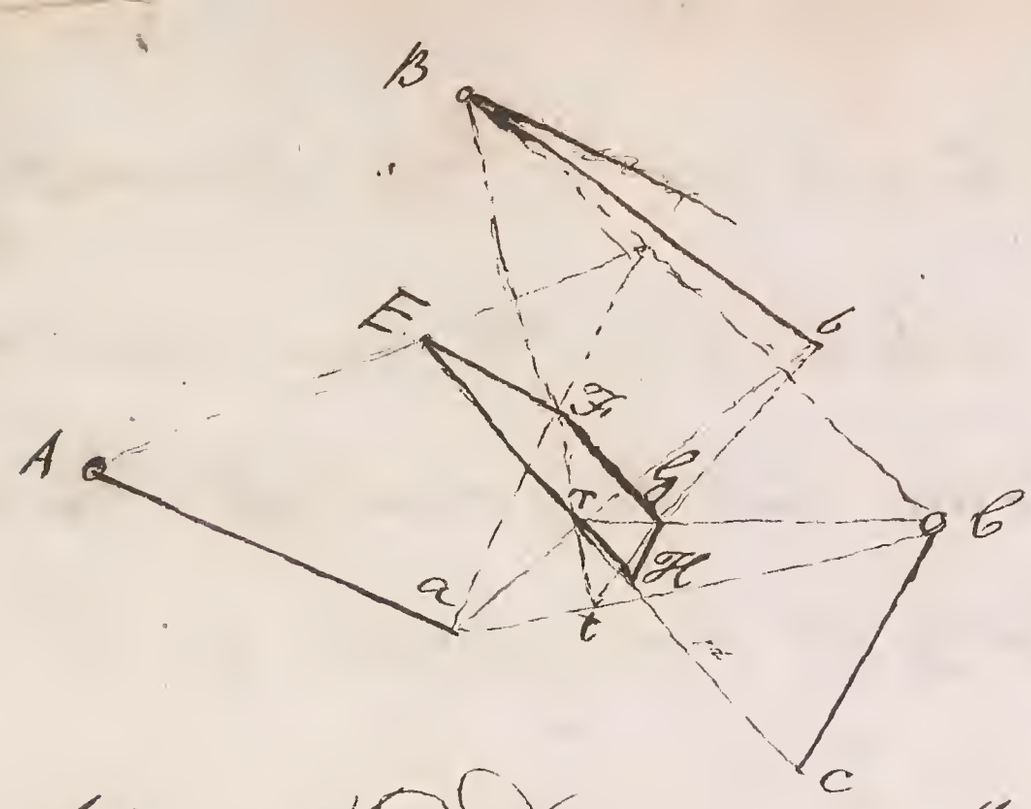
Let ABC be a triangle, draw AD & BE bisecting BC and AC, F is the centre of gravity, it is evident that the line AD cuts the whole into two equal parts, therefore the centre of the whole must be in that line, and for the same reason it must be in the line BE, it is therefore in their intersection The distance of the centre of gravity F from the angle A is to the whole line AD as 2 to 3 that is FD is to FA as 1 to 2; join DE that line is parallel to AB because BD, DC & AE, EC are equal to one another, therefore the triangles DFE & BFA are similar and DF is to FA (as DE to BA or) as EC to AC, that is as 1 to 2.

The centre of Gravity of a uniform texture sphere is its centre of magnitude and that of a cube or parallelipiped is the intersection of its diagonals. The centre of gravity of any irregular body is found by suspending one extremity of it by & marking the direction of gravity by a plumbline and causing it to pass through the centre, then another line being found in the same manner by suspending another end the common centre of gravity will be found in their intersection

If any number of bodies be impelled by any forces whatever, their common center of gravity will move in the same manner as those bodies would have done had they been all collected into that center and impelled by the same forces in the same directions

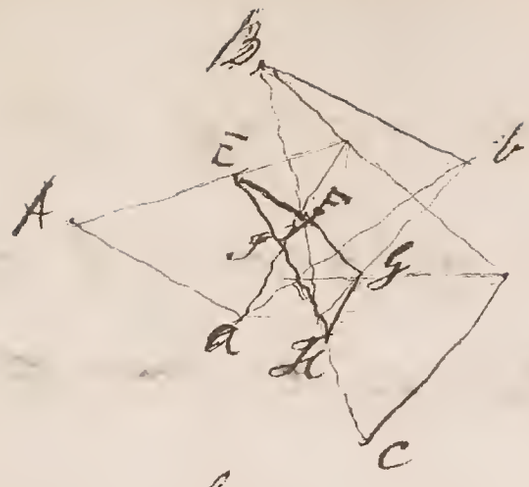


Let A, B & C be ~~any number of bodies at rest,~~ three bodies at rest let their common center of gravity ~~be~~ S be at E and supposing A to be taken away, let S be the center of the other two B & C; Now (B & C remaining at rest) let A be moved uniformly in a right line from A to a, at the same time S will have moved from E to F in a right line parallel to A a, and from the nature of the ~~center~~ of gravity F will be in the right line a D which is cut proportionably to A D; then as E, D is to E A as A to B + C componendo E D will be to D A ~~as A to A + B + C~~ that is E F will be to A a as A to A + B + C. In the same manner it may be proved that B be moved to b that I will move to G which will be found in the line b t joining b and the intersection of B F and a C, and that F G will be parallel to B b and be to it as B to A + B + C. And if C be moved to c G H will be parallel to C c and will be to it as C to A + B + C; and the spaces A a, B b, C c and F G, G H described in a given time will be the velocities of A, B & C and of the center S at the points E, F & G. Now suppose the bodies to return to their original situations A, B, C and let them be acted upon by forces in the directions A a, B b & C c with the velocities respectively by those lines. Then as S was moved thro F & G to H by the separate motions of A, B & C when they are all united it will move from E to H in the same time that it moved from E to F from F to G and from G to H; it will therefore be represented by E H.



Now suppose abd ~~D equal to the sum of the bodies~~ A, B, C be placed at the common center of gravity E; let it be impelled by the ~~same~~ force at A only, which will drive it in a line E F parallel to A a ~~and~~ with a velocity which will be to that of A as the body A to the body D that is as E F to A a, therefore D will describe E F in the same time ~~t~~ that A would have described A a; in the same manner if D, ~~be~~ at F, be impelled by the force at B it will move ~~it~~ to G in the same time t that B would have moved to b; and if it be ~~in~~ impelled by the force in C it will move ~~from~~ G to H in the same ~~the~~ i.e. its center of gr given time t; from which it follows that, if D, be placed at E and be impelled at once by the forces in A, B & C, it would be found at the expiration of the time t ~~time~~ in the point H that is it would describe the space E H in the given time. But the time t may be taken of any length, therefore the center of the body D will allways coincide with that of the system and will be found to ~~exist~~ in the same point at the end of any ~~give~~ time.

Cor. 1st. If the bodies A, B, C move uniformly in right lines whether their lines of direction be in the same plane or not, the center of the body D & X and consequently that of the whole system will ~~be~~ be at rest or move uniformly in a right line.



EF, FG, GH, HE

b Cor. 2. The distances ~~EF, FG, GH, HE~~ are as the

C motions of the bodies A, B, C & D. As D describes

EH in the same time that it describes Aa; the

velocity of D will be to the velocity of A as EH to Aa and the

quantities of motion as D × EH to A × Aa; but as EF is to Aa as A to

D, A × Aa will be equal to D × EF; and the motion in D to that

in A as D × EH to D × EF that is as EH to EF. In the same manner

it may be proved that if EH represent the motion of B

EF, FG, GH will represent the motions of A, B & C.

Cor. 3. Having given the motions of A B & C that of

their common center may be found, for drawing the lines EF,

FG, GH parallel to the directions, and representing the forces

A, B, C and joining EH, it is evident that EH will

represent the motion of the centre in the given line

Cor. 4th If the last point H coincides with the

first E it is evident that the center is at rest.

Cor. 5th As the line EH represents the motion of the

common centre of A & B whole system, so EG represents that

of the center of C only, FH of B & C only and if the parallelogram

EFHG be completed EI will represent the motion of the

center of A & C only

Cor. 6th External forces acting upon the system

affect the motion or rest of the common center of gravity in the

same manner as equal and similar forces acting upon

the body D would have affected the motion or rest

of center of gravity, and makes no difference in

whatever part of the system this external

force is provided it be equal and similar

given
ground at the

Cor. 7th Equal and and contrary forces acting upon a syste
have no effect upon the common centre of gravity whether
the be impressed upon the same or different parts of the
system.

Cor 8. As action and reaction are equal the actions of
different parts of the system upon one another do no
effect the motion or rest of their common center of grav

viz The 1st Law of Motion

Cor. 9th The Laws hold good with respect to the
motion of two or more bodies as with one, as the motion
the common
motion of both is estimated by that of their center of
gravity which will lie at rest or move uniformly
in a right line if not acted upon by some external
force.

1 o c o

The Motion of Animals.

Animals which move upon two feet are men and birds; the man performs the operation of walking by ~~stretching~~ ~~one leg upon the~~ bringing the centre of gravity of his whole body over one of his feet so as to be supported by it, and at the same time stretching the knee of that leg and placing the other before it, and so on; a man runs in the same manner, with this difference that in walking he has always one foot upon the ground and sometimes two, and in running he is sometimes entirely raised from it and never touches it with more than one foot at a time. A bird ~~runs in~~ some birds move by a sort of spring or leap others walk in the same manner as men; a man when standing is supported upon the bones of his two legs as upon two pillars; a bird supported by the muscles. If a man be sitting or a beast lying the centre of gravity must be over or beyond the feet upon which they are supported before they can rise.

When a horse or other quadruped begins to walk he stretches one of his hind legs so as to throw his centre of gravity forward, and raising that leg from the ground he places it in an other position and after that he steps on with the fore foot on the same side, and then stretching the hind leg on the other side he moves that as before and directly after it the fore leg on the same side with the last and so on when a horse trots he moves the two feet diagonally opposite at the same time; when he gallops he takes up his feet one by one and sets them down one by one; though some animals strike with their two fore feet nearly at once and their two hind feet in the same manner and he never have above two feet on the ground at the same time.

Animals with six or more feet begin with the hindermost and mo
those on the same side one after another, and then beginning with
the last of the other side they move them all as before.

A bird flies by the resistance of the air upon his wings
(rowing)
which, being placed so as to form an angle with the horiz
the ~~force acting upon the wing~~ reaction of the air may be resolve
into two forces one in a direction perpendicular to the horizon
and the other parallel to it; the one prevents the bird from
falling by its gravity and the other gives him his motion
forward; when a bird wants to ascend he raises his tail, on the contrary
when he wants to descend he depresses it down, for some pro
of the resistance acting upon the tail acts perpend
to the horizon and therefore turn the bird about his
centre of gravity (which lays between the wings) and point
head and consequently the direction of his motion is either up
down as the tail is raised above or below the level of his
body. A bird turns to the right or left by striking with the le
wing or right ~~wing~~ wing, with the other without moving the
other.

The swimming of a fish is performed by the reaction of
the water upon its tail with which it strikes the water and which
it bends so as to form a curve convex to the resistance of the wat
from this shape the reaction will act at an angle with the
body and may therefore be resolved into two forces; the one di
so as to move the fish about its centre of gravity, an
the other the other is to drive it forward, the first is correc
by the next stroke, in 2 strokes the only effective force is y propu
A fish is provided with fins on each side which enable it to turn
and which act ~~like~~ like oars.

Most fishes have a bladder which he can fill with air or empty when they please and by that can rise or sink or remain any where at rest at pleasure.

As the center of gravity of a fish is situated near the back if it would always swim with its belly upwards were it not for fins that it has on its back and belly which being expanded it is kept in its proper position from the resistance of the water against it.

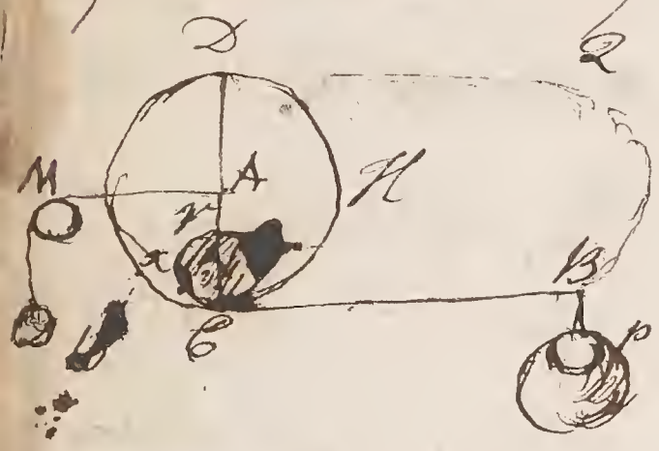
Four footed animals and birds swim naturally, as they require but a very small part of their Body to be above the water for breathing, they will therefore float without motion, the same motion which the use in walking or running will be sufficient to carry them along; birds being much lighter than water will likewise float. But a man who is nearly the same weight with the water and who is obliged to have his whole head above it can not learn to swim without a great deal of labour and practice.

The absolute power of every animal muscle must necessarily be greater than the power it has to support

~~Because the muscle acts upon the cubit very near the centre of motion of the elbow upon the~~

As the cubit of the arm may be considered as a lever moving about the centre of motion at the elbow, upon which the muscle acts so that the distance from the land which supports the weight from the center of motion it to the distance from the bone to which the muscle acts to the same center of motion is 20 to 1; so that if the weight supported by the hand be together with the weight of the cubit 20 pound, the strength of the muscle will be 600 pound,

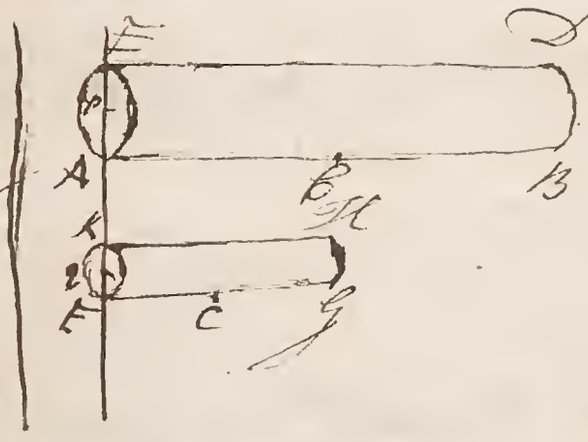
If a power tends to break a hollow body, it will have less effect than of the power of resistance was collected into one solid mass of the same weight as the former.



Let the the power P tend to break the hollow body D Q B C, the resistance is greater than if the same was collected into one solid body as C r.

As all the resistance of the body acts at their centres of gravity A & Y and that the circumference of D H C is greater than r x it is evident the the semidiameter of the one is greater than that of the other and consequently the power greater on the bent lever A C B. From this it appears that the bones of animals & quills of birds which are hollow have a stronger than if they were of the same weight and in one solid mass.

[The strength of similar bodies of equal densities is increased as the magnitudes



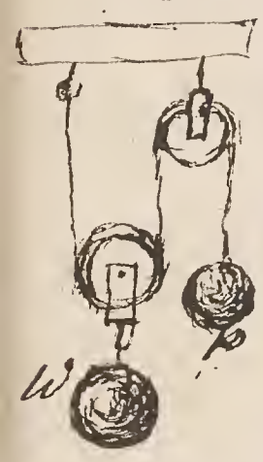
Let E B and K G be two similar cylindrical bodies in which the length A B E is double the length of the other K H; let these two beams adhere to the wall E F. Then it is evident that the weights will be as 2 to 1 and as the distances of the centers of gravity C & C are as 2 to 1 the powers to break off will be as 16 to 1; Now as the beams are supposed to be composed of the same substance, the number of fibres or the quantity of cohesion will be as the areas E A, K F that is as the lines E A, K F or as as 4 to 1, and (supposing the whole to be collected in the centers P & Q) because the distance is double as 2 to 1. therefore in general the cohesion increases in the duplicate and the power to break off in the

ratio of the lengths.

If a power supports a weight by means of a fixt pulley the power will be equal to the whole weight.

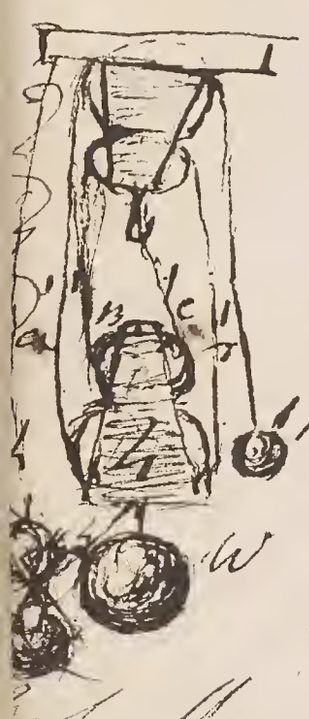
Let one of them be moved up or down, the string as the length of the is given;
the other will describe a space equal in the same time, (it is evident the weights will be equal
at any distance from the pulley the equilibrium will
be unaltered) therefore the weight and power are
be equal to one another.

If a power supports a weight by means of a moveable pulley, the power will be half the weight.



It is evident that if the power be moved, the weight will describe half the space, but
are in equilibrio move in a given time are
the weights of matter
to one another inversely as the spaces they describe
P will be to W as 1 to 2.

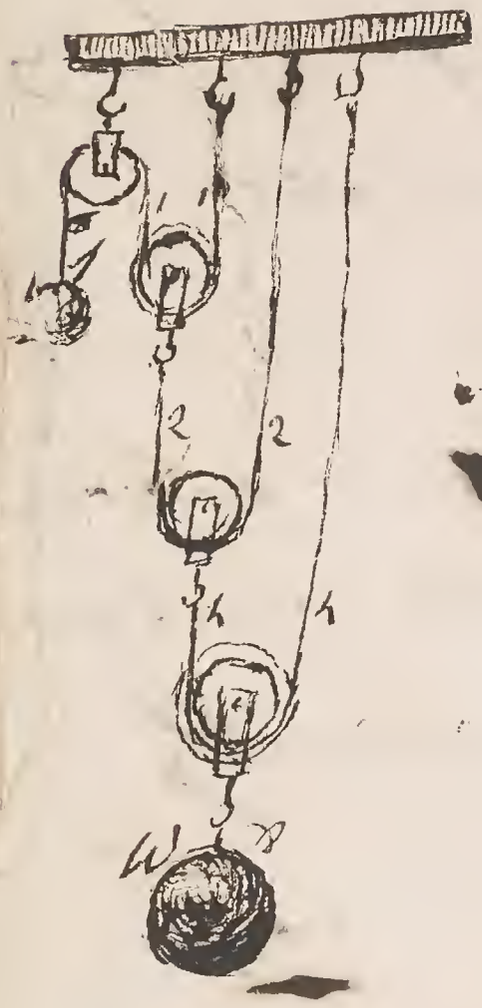
In a combination of pulleys which have only one cord, the power is to the weight as unity is to twice the number of moveable pulleys.

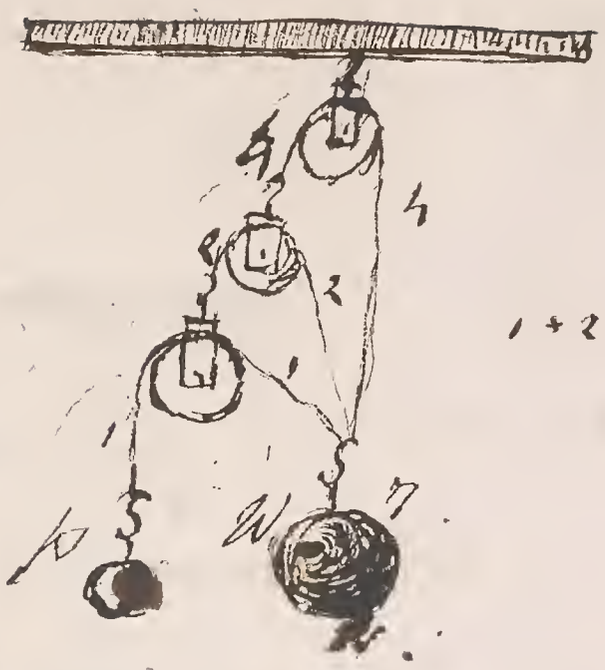


Suppose the powers to act so as to raise the weight
one inch, then as there is an inch taken from each
cords
of the lines A, B, C & X it is evident that the power must
move four, and as P is to W inversely as the spaces they
describe P : W : : 1 : 4 and in general P : W : : 1 : n° of moveable
twice the
pulleys.

In a combination of moveable pulleys in which each
has a seperate string, the powers will be to the weight
as 1 to that power of two whose index is the number of
strings

Let





1 + 2 + 4 = 7 = 2^3 − 1

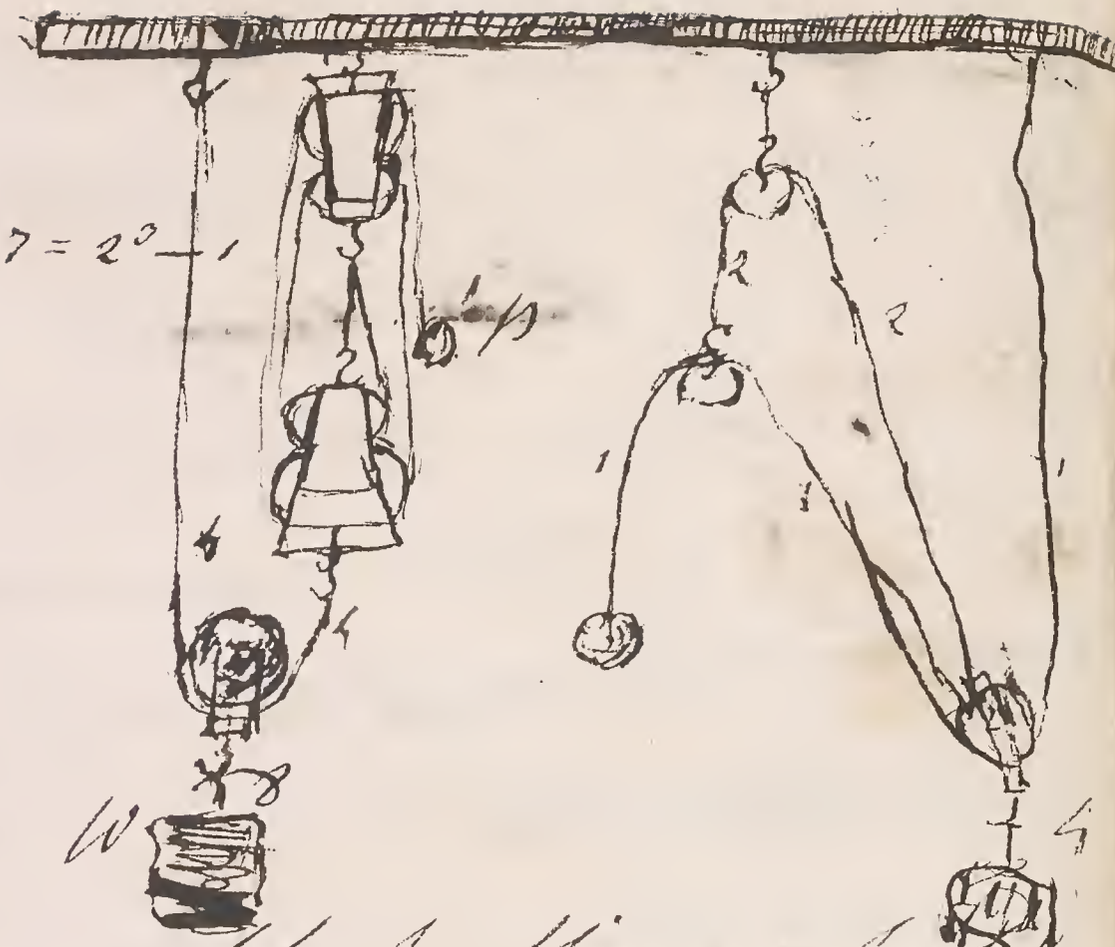


Fig: 1st In a combination of n moveable pulleys each of which has its seperate string fixed at one extremity, P : W :: 1 : 2^n

~~At the power is 1, each of the~~ the strings in each side of the first moveable pulley are ~~re~~ each of them ~~by~~ the Power or 1, the effect of the first moveable pulley to raise the next will be 2 the effect of those to raise the third will be 4 and so on till it comes to the weight, which will be ~~made~~ acted upon by 2^n from which it is to the first as the power of n whose index is the number of pulleys ~~1 to 1~~ to one; that is P : W :: 1 : 2^n

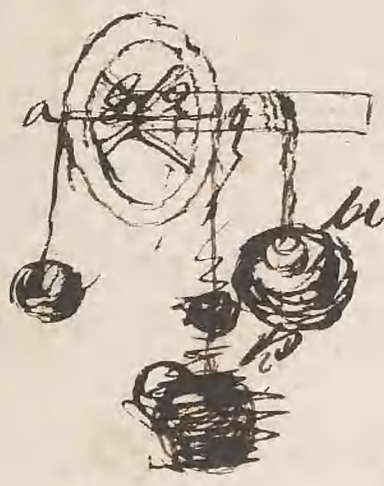
Fig: 2d In a ~~a~~ system of ~~pul~~ moveable pulleys where the strings all meet at W, P : W :: 1 : 2^n,

~~At the power is 1, the effect of the first pulley will be 2~~ of the next ~~pulley~~ and so on, and ~~the~~ the weight is acted upon by the same weight the effect of P upon W will be as one to the sum of the series 1. 2. 4 &c that is as 1 to 2^n − 1.

Call the power 1, then the effect of the first pulley will be 2 of the next 4 and so on; then as all these powers act upon W seperately, the effect of P upon W will be as 1 to the sum of the series 1. 2. 4 &c that is as 1 to 2^n − 1

Fig: 3 is only a combination of the

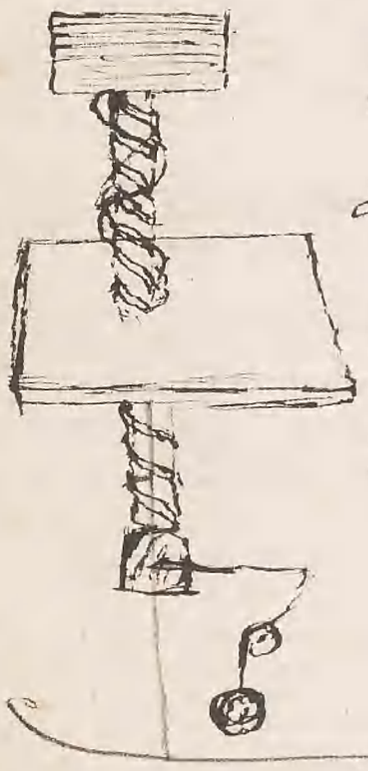
There will be an equilibrium on the ~~wheel~~ wheel and axle when the power is to the weight as the radius of the roller to the radius of ~~the roller to the radius of~~ the wheel



For if W be supported to act upon the lever A Q ~~it~~ with a distance from the centre of motion C equal to the radius of the roller, it may be considered as a lever of the first kind, and will be to Q as A C to A C, that is P : W :: roller : wheel

Let them be turned once round ... the power describes ... of the Circum: Radius : P. W :: Rr : Rw

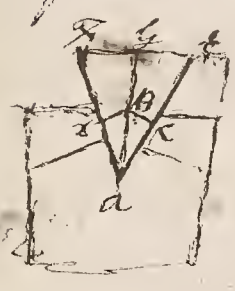
There will be an equilibrium on the screw when the power is to the weight as the distance between the two nearest spirals is to circumference described by the power.



As it is evident that the weight will rise a hight equal to the distance of one spiral from the other whilst the screw is moved round once therefore the P. W :: dist between 2 spi : Circumf

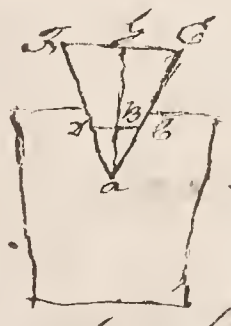
Then as the triangles C H G and A C B are similar G H will be to G C as A B to A C, and because the triangles F H G & C B D are similar G H will be to G F as B D to B C, the two compounded the ratios 2 g h : g f + g e :: (A D + D B or) A B : A C + C B,

When the direction of two forces acting on a wedge is perpendicular to its sides, the power is to the resistance of the sides as the back of the wedge to the sum of the sides.



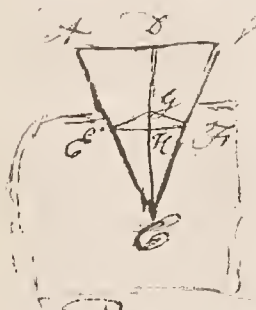
Let the force be represented by BA

Let BA represent the resistance of the sides, and let BC BA be the two forces acting at right angles to the sides; then in the triangles CAE, BAG, as the angles at C and G are right ones and the angle at A common, they are similar and BC is to BA as GE to CA, in the same manner it may be proved that BD is to BA as FE to FA that is the whole power is to the resistance as FE the back of the wedge to FA + AE the sum of the sides.



When the direction of two forces is parallel to the back the power is to the resistance as half the back to the height of the wedge: or as the whole back FE to twice the height

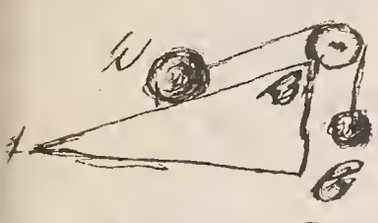
As DC, FE are parallel, this proposition may be proved in the same manner as the preceding.



When the direction of two forces, acting on a wedge is perpendicular to its sides, the power is to the resistances of the sides as the back of the wedge to the sum of the sides.

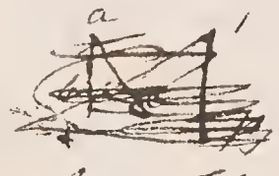
Draw CD perpendicular to the back, take any point C in the side AC and from it erect the perpendicular CG meeting CB in G, from G draw GF at right angles to the other side BC, join CF; then if the sides of the wedge be equal it is evident the CG, GF and CH, HF will be equal to one another Let CG and GF represent the whole forces or the resistance acting at right angles to the sides, then if CG be resolved into CH, HG and GF into FH, HG the effect of the whole power will only be as HG, as CH and FH which are equal and opposite destroy one another's effect

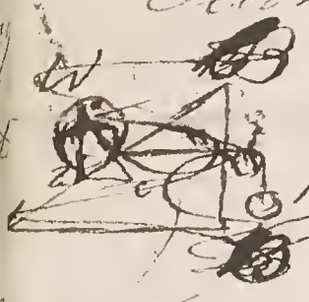
When a body is sustained on an inclined plane, & the power is ~~to the weight as the line of inclination to the whole~~ ~~direction~~



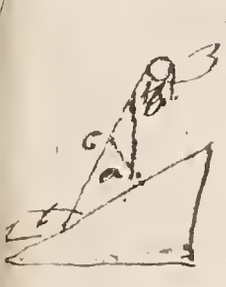
~~line, suppose the power to~~ acts in a direction parallel to the ~~inclined~~ length of plane, let ab represent ~~the~~

whole ~~weight~~ force in ab; let it be resolved into ac, bc the one at right angles to the plane and the other parallel to it; then ~~as the plane is fixt,~~ ac will be the reaction of the plane ~~has no effect~~, and W in the direction of the plane with the force bc will act ~~only as bc~~ which will therefore represent the power requisite to support W; then as the triangles abc, ABC are similar, ~~bc : ba :: BC : BA~~ that is power is to the weight as the height of the plane to the length.





Secondly, If the power act in a direction parallel to the base; let ab represent the whole weight, and let it be resolved into ac, at right angles to the plane, and bc parallel to the base, then as before bc will represent the power and because the triangles abc, ABC are similar bc : ba :: BC : CA that is the power is to the weight as the height to the base of the plane.

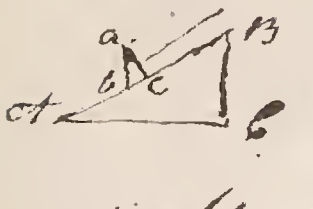


And in general

In whatever direction the power acts, P will be to W as the ~~cosine~~ sine of the angle of inclination to the cosine of that of traction. Let any line ab ~~in~~ perpendicular to the horizon represent the whole weight, ~~let~~ it be resolved into bc, in the direction of the power and ca at right angles to the plane, ~~then~~ it is evident that cb will represent the power as before, then in the triangle bca as the sides are as the sines of the opposite angles cb is to ab as sine angle a to sine angle acb or ac or (as ac & cba are equal to a right angle) to cosine angle cba; therefore in general ~~P : W :: cos. of Incl. : cos. of tract~~ but as the lines ba, ca are ⊥ to the lines BC, BA the angle containing them will be equal

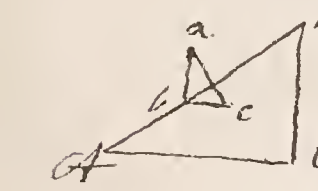
~~P : W :: cos of Incl. : cos. of tract~~

This last ~~proposition~~ case holds good in the two first.



In the first case; as the angle of traction is nothing (the direction of the power being parallel to the plane) its complement to a right angle will be a right angle, therefore its cosine will be the ratio of a right angle, or radius. Then P will be to W that is, cosine ang. of Incl: cos. angle Contact :: P : W.

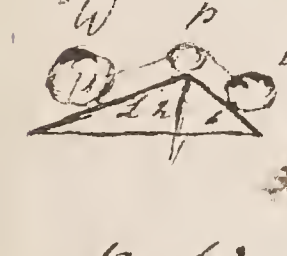
In the second case, as the direction of the power is parallel to



the base the angle of traction and inclination are equal to each other, therefore the power is to the weight as the sine to the cosine of the same ang. I.

From this it is evident that the power is at a minimum when it acts parallel to the inclined plane as the less the angle the greater its cosine, on the contrary it must be infinitely great to support the weight, when it acts at right angles to the inclined plane if the power acts at right angles to the Base it will be just equal to the weight, ~~but~~ for as it acts in a direction directly contrary to the weight it must be equal to it to support it

If two Bodies sustain one another on two inclined planes of equal the same height the will be to one another as the lengths of their respective planes

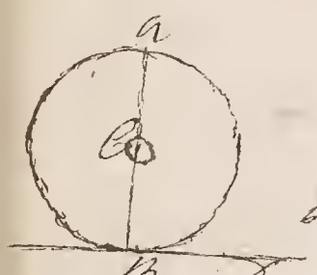


Let a power be applied at P so as to sustain W then P : W :: H : L and by applying the same power at P it will (be) W : P :: L : H, then by compounding the ratios W : w :: L : L.

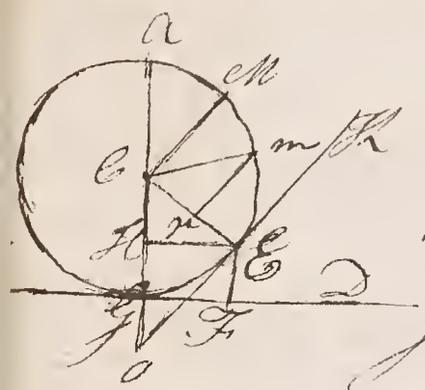
ntage

Of Wheel Carriages,

If the weight of a body and its roughness be given its friction & will be the same in whatever number of points it touches the plane on which it moves; as ~~any~~ is found by experiment.



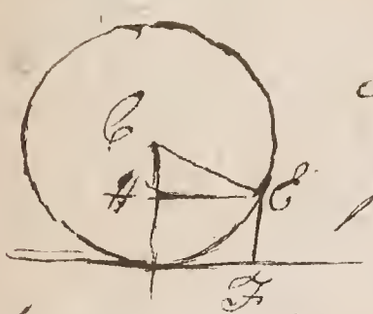
The use of a wheel is to diminish the friction. Let ABS be a wheel moving about the axle at C; first let it be carried along by the axle C so as not to touch the ground in any point, then as the Centre of Gravity of the wheel is supported by the whole the whole will move in a line parallel to that described by C; Now suppose the wheel to touch the plane BD in the point B, then on account of the friction B will be made to move slower than C & the other extremity of the diameter A, but it is impossible that A should move forward without B moving back, & consequently the wheel turns perpetually on its axis & the friction at B is referred by the circular motion to the axle. will be accelerated by the motion of C which meets with no opposition



Let PAM be a wheel moving upon the horizontal plane MD, let CE be an immoveable obstacle upon that plane, join CE, then the power acting at its centre requisite to pull the wheel over the obstacle will be to the weight of the wheel as the sine of the angle contained by the line CE and a line perpendicular to the horizon to the sine of the angle contained by that same line (CE) and the direction of the power; or (putting r for the radius of the wheel & l for the sine of the angle contained by CE and the direction of the power & h for the height of the obstacle) $P : W \times \frac{\sqrt{2rh - h^2}}{r - l}$.

First suppose it fastened to be immoveable, then it is evident that before the wheel is pulled over it must be supported by the power upon the point C so that none of its weight act upon the plane MD or which is the same thing that it be supported upon an inclined plane CE passing through the tangent OC & therefore the power will act at the greatest advantage in the direction CE parallel to

CA



If the obstacle is to be protruded let C E represent the whole pressure upon E let be resolved into C H acting in a direct line with it and H E acting at right angles against it, which last only can take effect the other part being lost; therefore the power is to the resistance as H E to E C or as the sine of the angle H C E to radius.

If wheels be to be drawn out of a hole they are to be considered as acting against an obstacle.

The spokes of a wheel may be considered as levers of the second sort diminishing the friction at the axle, upon which the power acts at the ground the weight at the circumference of the axle and whose centre of motion is the centre axis of the axle.

Wheels are dished, that is their spokes are inclined outwards, to make the wheel more nearly perpendicular to the horizon when the road happens to be inclined or when one wheel falls into a rut whilst the other is supported.

If the axle be bent at the extremities the base is diminished the carr is more easily overturned; as is evident

The use of one piece of wood for the felly is to make the circumference of the wheel everywhere equal which. therefore will bear a gre weight, the pressure being spread over the whole; whereas when the wheel is composed of several pieces it is confin to a small space and will therefore act at greater ad to the axis

Advantages of High Wheels

Sink less in the dirt or in holes; because they have less curvature.

Require less force to pull them over obstacles or through holes. Because the greater the radius the less the angle H C E, and its sine.

Not liable to wear out. If there be 2 wheels whose radii are a 1 to 2 and that they describe equal spaces in the same time, the small wheel will make two revolutions for one of the great one and as each point in it touches the ground twice for one of the great wheel it will wear out sooner

Advantages of Low Wheels

More convenient to turn with & less liable to overtur through

Better adapted for the animal to exert his strength by throwing his weight

In going up hill they make the draught easier be the centre of gravity lies lower and gives the animal a firm footing by bracing upon his back. & the line of draug becomes more parallel to the plane

In going down they accelerate the motion of the carri less than high wheels

High wheels are calculated for bad roads or plains

Low Wheels for good roads and hilly country.

Of the collision of bodies.

The quantity of motion which is collected by taking the sum of the moments in the same direction & the difference of the moments in opposite directions is not altered by the actions of bodies upon one another. As has been proved in treating of the motion of the center of gravity of a system of bodies.

When inelastic bodies meet their common velocity after impact is equal to the sum of the moments divided by the sum of the bodies. Let the motion be estimated in the direction A moves in. Let A & B be the bodies & let a & b be their velocities; then Aa + Bb will be the sum of the moments before impact which by the preceding will be the same after and the velocity be common and as the bodies are non-elastic, having no force to separate them, they will move on as one body, with a motion equal to the sum of the moments before the stroke, equal to the sum of the bodies being divided by the sum of the bodies will evidently give the velocity with which they move.

When two non-elastic bodies meet, the sum of the bodies is, to the striking body as the difference of the velocities before the stroke to the velocity gained by the body struck. $A+B : A :: a \pm b : f$ or $f = \frac{A \times a \pm b}{A+B}$

If A & B move in the same direction it is evident that f will be equal to $\frac{Aa+Bb}{A+B} - b$ and if in opposite directions $f = \frac{Aa-Bb}{A+B} + b$

$f \times A+B = A \times a \pm b$, $A+B : A :: a \pm b : f$, therefore $\frac{A \times a \pm b}{A+B}$ is the velocity and $\frac{B \times A \times a \pm b}{A+B}$ the motion gained by B or lost by A the motion of A after impact is A into the common velocity $\frac{A \times Aa+Bb}{A+B}$ and that of B after impact $\frac{B \times Aa+Bb}{A+B}$. When B is at rest, the terms involving B vanish.

In perfectly elastic bodies the change of motion is double of what it is in non-elastic, Therefore $A+B : 2A :: a \pm b$ or $by = \frac{2A \times a \pm b}{A+B}$ the velocity gained by B, which being multiplied into will give the motion gained $\frac{2AB \times a \pm b}{A+B}$ therefore the whole motion of B will be $\frac{2AB \times a \pm b}{A+B} \mp Bb$ and its velocity $\frac{Bb - AB + 2Aa}{A+B}$ the motion of A after impact will be $Aa \mp \frac{2AB \times a \pm b}{A+B}$ and its velocity $\frac{Aa - Ba \mp 2Bb}{A+B}$.

If A & B are equal they will move after impact with interchanged velocities. First suppose them to move in the same direction when the velocity gained by B at A is $\frac{2A \times a - b}{A+B}$ or (because $A+B = 2A$) $a - b$ then the velocity of B after the stroke will be $b + a - b$ or a, the velocity of A before the stroke and the velocity of A after impact will be $a - a - b$ or $+b$ the velocity of B before the stroke, ~~or the motion was communicated in the direction of A.~~ In the same manner it may be proved if they move in opposite directions, that the velocity of A after the stroke will be $a - a + b$ or $-b$ and the velocity of B after the stroke $-b + a$ or a; if B be at rest before impact it will move on with the velocity of A and A will be at rest itself; that is the velocities will be interchanged that of B being nothing before impact.

If an elastic Ball strikes upon several equal elastic balls lying in a right line at rest, the remotest will fly off with the velocity of the first mover and all the others will remain at rest. For if any force act upon the first so as to make it strike the second it will impell it (the second) with that force and be at rest itself; the second will strike the third and so on till it comes to the last which having nothing to resist it will move on with the velocity of the first.

If motion be communicated from a smaller elastic body to a greater by means of several intermediate bodies continually increasing in magnitude and each lying contiguous in a straight line & at rest, all but the last will be reflected & each ~~each~~ with a velocity greater than the ~~velocity~~ velocity of the preceding body; but the last will move forward with a motion equal to that of the first ~~mover~~ mover added to the sum of the motions of the other bodies; And this motion will be the greatest possible when the bodies are in geometrical progression.

Let A, B, C ~~&c~~ be three bodies situated as in the Proposition ~~of~~ first let them be to one another as 2, 4, 8, and let A be impelled with a force that may move it with a velocity as 1 and motion as 2 then. $B \times \frac{2Aa}{A+B}$ = motion communicated.

~~motion communicated to B $\frac{32}{9}$ Vel. $\frac{16}{9}$~~

~~to C $\frac{32}{9}$ Vel. $\frac{8}{9}$~~

~~motion of A 2 velocity 1~~

~~Motion communicated to B $\frac{20}{9}$~~

~~to C $\frac{320}{91}$~~

A, B, C as 2:4:8

motion of A	2	Velocity 1	A's reflected motion	$-\frac{2}{3}$	vel: $-\frac{1}{3}$
of B	$\frac{8}{3}$	$\frac{2}{3}$	B's	$-\frac{8}{9}$	$-\frac{2}{9}$
of C	$\frac{32}{9}$	$\frac{4}{9}$			

$\frac{32}{9} - \frac{2}{3} - \frac{8}{9} = 2$ the motion of A

$\frac{32}{9}$ the motion communicated to C is greatest in this case, where bodies are in geometrical progression, tho in any other, Less impulse motion to be communicated from A to C through B as C 3 or as 5; tho in the first case the motion communicated will be $\frac{192}{55}$ or $3\frac{27}{55}$ and in the other $\frac{320}{91}$ or $3\frac{47}{91}$ and reducing them to the same denominator

1st motion communicated through B = 3 : $3\frac{22113}{45045}$

= 4 : $3\frac{225013}{45045}$

= 5 : $3\frac{223760}{45045}$

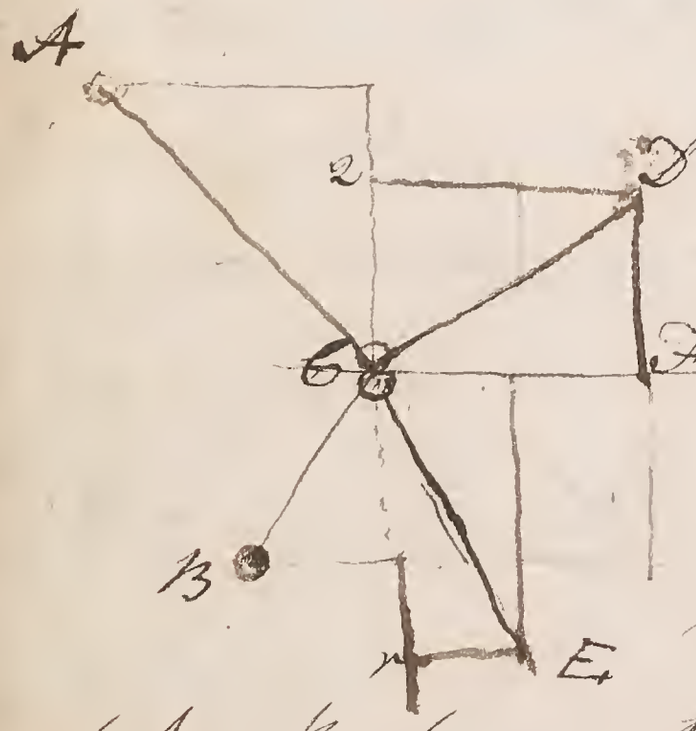
From which it appears that the motion communicated through B = 4 or when the bodies are in geometrical progression, is greater than in any other case

If one of the impinging bodies perfectly hard and the other elastic the laws of collision are the same as if they were both elastic. For when there is one body hard and the other hard the elastic body is indented with the whole force of the stroke and in the other case the same force is divided between two which will therefore produce the same effect.

If an elastic body strikes perpendicularly upon an immoveable hard or hard body it will rebound with the same velocity it came.

When an elastic body strikes obliquely on an immoveable obstacle and is reflected, the angle of incidence is equal to the angle of reflexion.

Let the body A strike the immoveable elastic or hard obstacle B D in the oblique direction A T and with a force represented by that line; the angles A T C & B T D are equal. Let the force A T or C T be resolved into the forces A E & E T the one perpendicular to a tangent at T, take E B or T D equal and opposite and as the force of expansion is equal to that of impact A E or C T, T E will express the perpendicular and E B or T D the horizontal force with which A is reflected, therefore A being acted upon at once by the two forces T E & T D it will move in their diagonal T B; then as B D, A C & T D, T C are equal to one another and the angles at D & C right ones the force T B will be equal to the force A T and the angle B T D to the angle A T C. Q.E.D.



If the impulse of one elastic body upon another in oblique it must be resolved into two impulses, one perpendicular and the other parallel to the tangent at the point of collision. Let A & B be two equal elastic bodies which meet in the point C through which let a tangent plane be drawn; let the forces of A & B be resolved each of them into two one at right angles the other parallel to the tangent plane, then as the perpendicular forces of the bodies are opposite in direction they will be reflected with interchanged velocities and the parallel forces will remain the same as before the bodies will move in the diagonals C D & C E of the parallelograms and interchanged perpendicular forces.

The motion to be estimated in the direction A moves in

Let A . B be the bodies Non-elastic

α . β their velocities

Case 1st Moving in the same direction.

Motion before impact $A\alpha + B\beta$

Common Velocity after impact	$\frac{A\alpha + B\beta}{A+B}$
Velocity communicated to B	$\frac{A \times \alpha - \beta}{A+B}$
Motion lost by A. communicated to B	$\frac{B \times A \times \alpha - \beta}{A+B}$

Motion of A after impact

$$\frac{A \times A\alpha + B\beta}{A+B}$$

Motion of B after impact

$$\frac{B \times A\alpha + B\beta}{A+B}$$

Case 2? Non Elastic in contrary Direction

Motion before impact $A\alpha - B\beta$

Common Velocity after impact.	$\frac{A\alpha - B\beta}{A+B}$
Velocity communicated to B.	$\frac{A \times \alpha + \beta}{A+B}$
Motion lost by A or communicated to B	$\frac{B \times A \times \alpha + \beta}{A+B}$

Motion of A after impact $\frac{A \times A\alpha - B\beta}{A+B}$

Motion of B after impact $\frac{B \times A\alpha - B\beta}{A+B}$

If B is at rest all the terms involving β vanish.

Case 3d. Elastic moving in same direction

A & B the Quantity of Matter

a & β the Velocity,

Motion always estimated in the direction of A.

Velocity gaind by B $\frac{2A \times \alpha - \beta}{A+B}$

Motion gaind by B $\frac{2AB \times \alpha - \beta}{A+B}$

Motion after impact of B. $B\beta + \frac{2AB \times \alpha - \beta}{A+B}$

Motion after impact of A. $A\alpha - \frac{2AB \times \alpha - \beta}{A+B}$.

Velocity of B after impact $\frac{B\beta - A\beta + 2A\alpha}{A+B}$

Velocity of A after impact. $\frac{A\alpha - B\alpha + 2B\beta}{A+B}$.

Case 4th — Elastic in contrary Directions

Velocity gaind by B $\frac{2A \times \alpha + \beta}{A + B}$

Motion gaind by B $\frac{2AB. \times \alpha + \beta}{A + B}$

Motion after impact of B. $+ \frac{2AB \times \alpha + \beta}{A + B} - B\beta$

Motion after impact of A. $A\alpha - \frac{2AB \times \alpha + \beta}{A + B}$.

Velocity of B after impact $\frac{A\beta - B\beta + 2A\alpha}{A + B}$

Velocity of A after impact $\frac{A\alpha - B\alpha - 2B\beta}{A + B}$

my

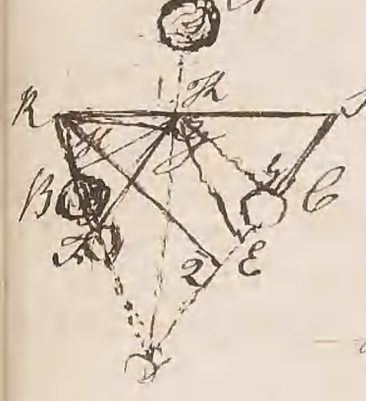
If motion be communicated from a smaller elastic body to a larger by means of several intermediate bodies continually increasing in magnitude & each lying contiguous in a straight line & at rest; all but the last will be reflected & each with a velocity greater than that of the preceding body; but the last will move forwards with a motion equal to that of the first moved & the sum of the motions of the other bodies. And this motion will be the greatest possible when the bodies are in geometrical progression.

Hamilton.

~~The Mechanical powers demonstrated by the composing~~

~~Theorem~~

If three forces be kept in equilibrio on a line, whether straight or crooked, any two of them are to one another inversely as the perpendicular ~~directions~~ distances to their lines of direction from the point at which the third force acts.



Let A, B & C be three forces in equilibrio upon the inflexible line RS, ~~the first forces are to one another inversely as the perpendiculars~~ any two of them. First let the lines of direction of B & C meet in the point D. Then as the force A is equal to the forces B & C compounded, and counterbalances them, it will act in the direction of ~~the~~ of the diagonal RD, drawn from the point of intersection D to the point R at which A acts; and may will be represented by it; from the point R draw RB & RG perpendr. and GE, RF parallel to the directions RF, GC; Then as the three forces are in equilibrio, they will be represented by the sides and diagonal of the parallelogram FRED, therefore B is to C as FD to DC or) FR that is as the sine of the angle (FRD or of its supplement to two right angles) FDE to that of the angle FDR; Then making RD radius RG & RH are the sines of those angles, therefore B is to C as RG to RH the perpendiculars from R.

Now to compare A & C together; It is evident, from the first part, that A is to C as RD to HD that is as the sine of the angle (RFD or of its supplement to two right angles) FDE to that of the angle FDR; Then drawing the perpendiculars RQ & RG they will be the sines of the angles and, as before, A will be to C as RQ to RG.

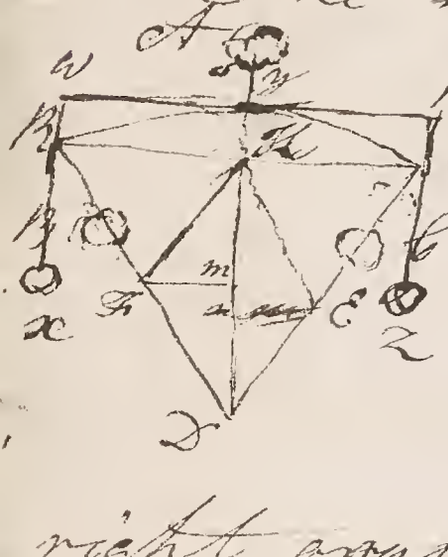
In the same manner it may be proved that A is to B as the perpendiculars from S to RF and AR.

~~Robert Hamilton~~

by m Case 2. When ~~the~~ two of the forces act parallel one another

each
to and consequently to the third.

The construction remaining as before, only that if the



lever be crooked, let its extremities K & L be joined
by the line K L
and let the intersection of that line with the direction
of A be taken for the point K. Draw FM at
right angles to KD; then as B is to C as FD to FK, if FD
be resolved into FM, MD and FK into KM, MF the forces in the
direction FM being equal and opposite will destroy each other
and B will be to C as DM to MK; then as those lines represent
the only parts of the forces B & C which act in the direction AD
and that they are together equal to KD, the force in A;
two forces acting at K & L in directions parallel to AD, and
which are in equilibrio with A may be represented by them,
let x & z be ~~represent~~ those forces, then x is to z as DM to MF
or as DF to FK or LK to KK or (by drawing WP at right
angles to the parallel directions) as BG to GW.

And as x & z are together equal to the force in A, and that
x & z are represented by GP & GW the whole force in A will
be represented by WP; therefore z will be to A as WG to WP
and x to A as BG to PW. Therefore if three forces be in
equilibrio on an inflexible line whether straight or crooked,
any two of them will be to one another inversely
as the perpendicular distances of their lines of Direction from
the point at which the third power acts.

By placing a fulcrum at any of the three points, this
inflexible line will become a lever of the first, second or
third order.

When there is an equilibrium on the wedge, the power is to the resistance as the height, to the line drawn from the middle of the wedge, parallel to the direction of the resistance, and terminated by the side of the wedge.

Let the isosceles triangle ABC represent a wedge which is acted upon by a force in the direction DE at right angles to the base and bisecting it and the angle at C; draw DR & DI parallel to the resistances.

First let the resistances act at right angles to the sides, then if the parallelogram DRQI be completed, the diagonal DQ may represent the forces acting upon the back of the wedge, as the sides in the directions of the resistances may represent them; then if RI be joined it is evident that DQ will be bisected in G, and DR will be to DE as DR + DI to DQ, therefore DR may represent the whole resistances and DG the whole force Then as the triangles RDE, QDR are similar DG is to DR as DR to DC the power is to the resistance

Now suppose the resistances to act upon the sides in the oblique directions DO DM then it is evident by resolving these forces, that their perpendicular resistances are equal to RD, DI and consequently the force necessary to overcome them may be represented by the diagonal DQ, or the whole perpendicular resistance by RD and the force by DE, then by drawing the perpendicular RP it is evident that DP may represent the whole resistance in the direction DO;

In the right angled triangles DRP, DRG as RP & RE are drawn at right angles to the base, the rectangles PDO & EDG being equal to the square of RD are equal to one another, therefore DE is to DP as DO to DG that is, the power is to the resistance as the line drawn DC - 26 D

VIII.

A Weight of 300 ℔ supported by a Screw the distance between 2 Spirals ½ an Inch, the circumference of the Power 5 feet } The Power — 2½.

IX.

Two Unelastic Bodies A = 1 03 B 4
Moment of A after impact : Moment before impact : 1 : 3 } Velocities before Impact. 6. 1

X.

Two Elastic Bodies A = 3 03 B = 9 Velocity of A 0 of B 2 } Velocities after impact 6. -1

XI.

Four Elastic Bodies A B C D The Velocity of A = 1 } Velocities & Reflections
A strikes B At rest B strikes C at rest & C D at rest } 1, ⅔, 4/9, 8/27 } -⅓, -2/9, -4/27

XII.

invariable Product of two Quantities } Their when

XIII

Two Elastic bodies (8) and 02 required the body which being placed between would communicate the greatest motion to the extremity. (8) (x) (2) x = 4.

XIV.

The Time of a Stones fall down a Well 3 Seconds } Depth

XV

A body is projected 108 Yards high } The Velocity of Projection

XVI

A body acquires a Velocity of 320 feet in a second } The Time The Space

XVII

A body came to the ground in 3" from a Tower 400 feet high. With what } Velocity which it was projected

XVIII

A Drop of rain was observed to fall two 112 Yards in the second before it reached the ground } The height of the cloud from which it fell. — + 645. 1/3

XIX

A Bullet is shot from a Tower with a velocity which carries it 10 feet in a second ——— } The space it will have descended in a Minute. yards 194 00

XX.

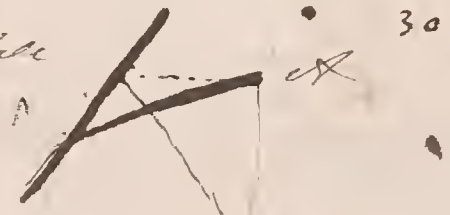
An Explosion was made in the air & the sound of it reached the ear exactly as a globe of fire came to the ground. Sound flies 1142 ft. in a Second. } Distance of the Cloud & the time of falling. — Gun. 71 3/8 Gal. 81510 1/4 +

XXI

A body was projected from the Top of a Tower 1885 ft. high with a Velocity which carries it 10 feet in a Second every instant that another body was thrown up from the bottom with an assigned Velocity of 100 ft in a Second. } The time that would elapse before their meeting and the point of the tower where they would meet. — 73.9536 1811.0464

XXII

Given a point & plane in position to find when the time of descent from A to the plane is a Minimum & in what direction it must fall } Parallel. Side of □ 30 Side of Hex



XXIII

The greatest Random of a Projectile is 2666 Yards. } Amplitude at an angle of 30.16 Parameter. 2325 4332

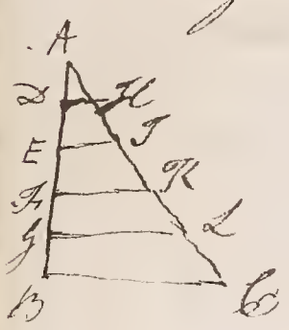
XXIV

The longest time of Projectile flight is 12 Seconds } The angle which it must be known in to be 6 Seconds going

A Chandelier in a Church recently rocked performs a vibration in 3 Seconds } 30. the height of the room

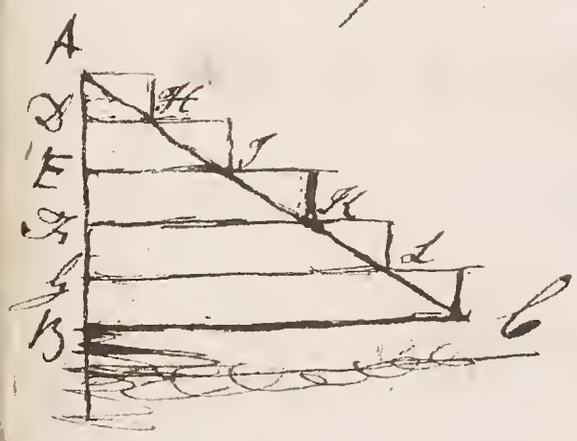
Of the perpendicular ascent and descent of Bodies

When a body is put in motion by a continued uniform force, the velocity acquired will be as the time of its motion.

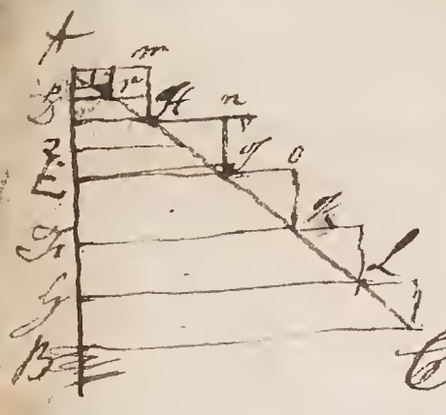


Let A B be the time in which a body falls by the force of gravity, the velocity acquired by it at any period of that time will be as the time it has been in motion; Let A B be divided into the equal parts or moments A D, D E, E F &c. let D H, at right angles to A B, represent the velocity it has acquired in the first moment A D; at the end of the second moment D E it will have both the first acquired velocity D H and that acquired in the moment D E, and, as the force of gravity acts equally in the first and second moments, E I, the second acquired velocity, will be double D H the first, or in the least instant the velocity F K will be treble the first, hence, in this case, the last acquired velocities are as the times. In the same manner it may be proved that E K is to E I as F L &c. &c.

The space described by a falling body is as the square of the time or of the last acquired velocity.



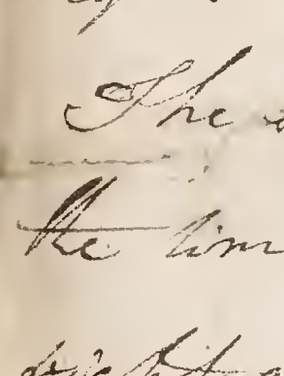
Let the body A fall by its own gravity in the time A B, the space described by it will be represented by the triangle A B C; For let A B be divided into any number of equal parts as A D, D E, E F &c and let the force act by a fresh impulse at each of these times; Let the velocity with which A moves in the first time be D H, then the space described in the time A D will be under the time and velocity the rectangle A H, and as I E the velocity in the second time D E, is double the first from the 1st & 2d Impulses together the space in that time will be the rectangle A I, double the first in the same manner it may be proved that the spaces in the times E F, F G, G B are represented by the rectangles E K, F L, G I &c.



Therefore the whole space described will be represented by the sum of those rectangles; now suppose the times AD, DE &c to be bisected in the points P, R &c and let A be supposed to move uniformly in those times as before, then as AP is half of AD the velocity PP will be half of DH and the space described in the time AD will be the rectangle AP and that in the time PD the rectangle DH therefore the space described in the time AB will be the sum of the rectangles Ap + PH &c; if the lines AP, BD &c be again bisected & triangles above the line AC will be still smaller; and if the line AB be divided into infinitely small parts, or (which is the case in the action of gravity) when the force acts constantly, they will vanish and the space described by A in the time AB will be expressed by the triangle ABC and the last acquired velocity by the side BC. And as similar triangles are to one another as the squares of their homologous sides the space described in the time AB by the body A will be as the square of AB or BC, that is of the time or last acquired velocity

The space described will be equal to half the product under the time of descent & the last acquired velocity. As ABC is a right-angled triangle it is evidently equal to half the product under its sides containing the right-angle.

The spaces described in equal successive parts of time computing from the beginning of the fall are as the terms of the arithmetic series 1, 3, 5, 9, 11, 13 &c.



Let the line AB be divided into the equal parts AD, DE &. Then the spaces described in those times are as the triangles ADG and the figures DH, EI &c. Draw Gp, Hm, In parallel to AB, join GEHBS; then it is evident that the figures DH &c are cut into triangles all equal to ADG; in DH there are 3 in EI 5 in IK 7 and so on. Therefore the spaces are to one another as 1, 3, 5, 7 &c

When a body ascends it is uniformly retarded, and the time of the motion will be as the first velocity and the space described as the square of the velocity or as the square of the time ascended

The spaces described in equal successive parts of time are in arithmetic progression.

The height to which bodies rise are as the squares of the first velocities.

Which are the converse of the former propositions and may be proved in the same manner.

From these propositions the equations for solving problems about falling bodies are found. putting S for the space described, or for a body falls through 16 feet in a second; v for the velocity & T for the time of motion. $S = \begin{cases} 16t^2 \\ \frac{v^2}{64} \end{cases}$ $V = \begin{cases} 32t \\ 2\sqrt{16S} \end{cases}$ $T = \begin{cases} \frac{v}{32} \\ \frac{\sqrt{S}}{4} \end{cases}$

The time of a stones fall down a well is 3 seconds } Depth $S = 16t^2 = 144$ feet.

A body is projected 108 yards high } the velocity of projection. $V = 2\sqrt{S} = 144$ feet $= 48$ yards in a second

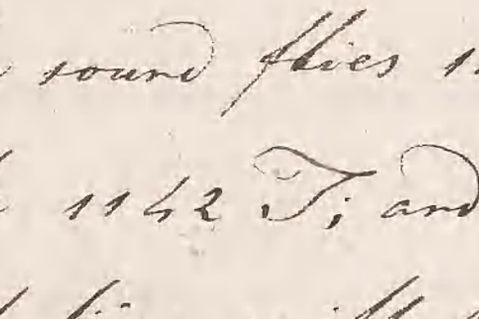
A body acquires a velocity of 320 feet in a second } The time $T = \frac{v}{32} = 10$ $S = \frac{v^2}{64} = 1600$

A body came to the ground in three seconds from a Tower 400 feet high } velocity with which it was projected. In the body would have fallen through 144 feet in 3" by its own gravity only, the velocity with which it was projected will be $400 - 144 = 256$ feet in a second

A drop of rain was observed to fall through 112 yards in the second before it reached the ground. Required the height of the cloud from which it fell.

Let DB represent 1 second, then the space described in that time will be the figure $DECB$; Draw EF at right angles to BC, it will be equal to DB, and may likewise represent a second, Therefore the space EFC will be 16 feet and DF 320 $\therefore$ the velocity BF and $BC = 352 = V$. $S = \frac{v^2}{64} = 1936$ feet $= 645\frac{1}{3}$ yards

A Bullet is shot from a tower with a velocity which carries it 100 feet in a second. Required the space it will have described in a minute.



$T = 582000$ feet or 19400 yards.

An explosion is made in the air, and the sound of it reached the ear exactly as a globe of fire came to the earth. Required the distance of the cloud and the time of falling.

As the sound flies 1142 feet in a second the space described by it will be $1142\,T$, and by the proposition the space described by the globe of fire will be $16\,T^2$ then,

$$16\,T^2 = 1142\,T$$

$$16\,T = 1142$$

$$T = \frac{1142}{16} = 71\tfrac{3}{8} \text{ seconds} = 1'\,11\tfrac{3}{8}''$$

$$S = 16\,T^2 = 81510\tfrac{1}{4} \text{ feet.}$$

A body was projected from the top of a tower 1885 feet high with a velocity which carries it 10 feet in a second at the very instant that another body was thrown up from the bottom with a velocity of 1000 feet in a second; Required the time that would lapse before they met & the point of the tower at which they would meet.

$$\text{Space descended} = 10\,T + 16\,t^2$$

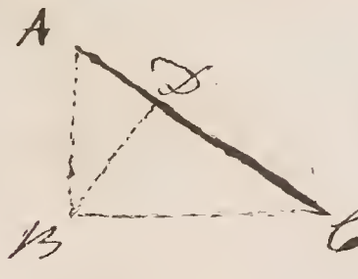
$$\text{Space ascended} = 1000\,T - 16\,t^2$$

$$\text{Therefore } 1010\,T = 1885.$$

$$T = 1.86633 \text{ \&c}$$

Of the Descent of bodies down inclined Planes

The force whereby a body endeavours to descend down an inclined plane is to the force of gravity as the height of the plane to the length.



Let A C be an inclined plane, ~~a body~~ the force with which a body endeavours to descend from A to C is to the force of gravity as A B the height to A C the length of the plane. As the force of gravity acts in the direction A B, if it may be represented by it, let it be resolved in the forces A D & B D the one parallel and the other at right angles to the plane A C, then as the effect of D B is lost by the reaction of the plane, the force in the direction A C will be represented by the line A D, therefore the force of gravity, is to the force tending to carry it down the inclined plane, as (A B : to A D or as) A C : to A B, the length : to the height of the plane. G : F :: L : H or $F = \frac{G \times H}{L}$ or $D = \frac{G \times H}{L}$ as G is given. F $\frac{H}{L}$

Since the ~~acceleration~~ force down an inclined plane has a given ratio to the whole force of gravity, it is likewise a ~~uniform acceleration~~ constant force & upon any given plane the velocities are as the times and the descents whatever was demonstrated of bodies falling freely may be applied to their falling down inclin'd planes as the squares of the velocities or times ~~except~~ only the velocities acquired in given times will be less. The universal expression for the velocity being $T \times I$.

2 Upon all kinds of planes the last acquired velocity is as the square root of the height from which the body has fallen.

3 The time of descent is as the length of the plane directly and the square root of the height inversely.

All which may be deduced from this expression.

~~F. (G) L :: I.~~ second $T^2 \mid H$ } given F. } $V : T$
that is the Plane. }

~~I : F × T^2 &c.~~ $I : T^2 : V^2$

$V : F \times T$

2° $\{$ $V : \sqrt{H}$ { Hence a body descending down an inclined plane will have acquired the same velocity as if it had fallen freely the same perpendicular height

$V : \frac{H}{L} \times \frac{L}{V}$

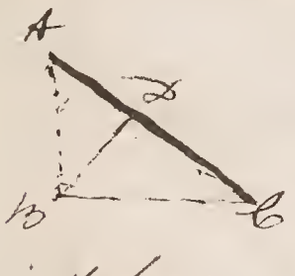
$V : \frac{H}{V}$

3° $\{$ $T : \frac{L}{V} : \frac{L}{\sqrt{H}}$

$V^2 : H$

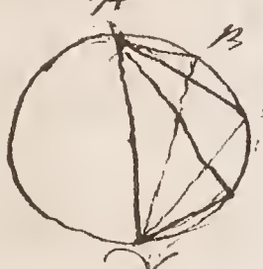
$V : \sqrt{H}$ Gen. for $L : VT : : T^2$

A body will come to a right angle upon an inclined plane in the same time that it descends down the perpendicular.



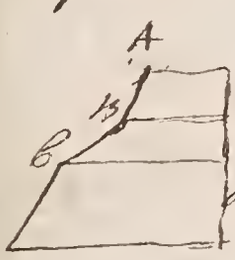
For A body will descend from A to D and from A to B in the same time when the time is given As F. & A B and A D are as F & L respectively represent the forces. Therefore

A body will descend through the diameter of a circle or any of its chords in the same time; and the relative gravities & the last velocities acquired are as the lengths of the chords described.



Let A B C D be a circle, a body will descend through its diameter A D perpendicular to the horizon) or any of its chords A B, A C &c in the same time; for draw the lines D B & D C, it is evident that A B D & A C D will be right angles and consequently a body will describe the spaces A B, A C &c and A D in the same time. And as the time is given F and V will be as L.

If a body descends with a given motion along any number of contiguous planes, it always acquires the same velocity in falling from the same height, whatever be the lengths or inclinations of the planes.

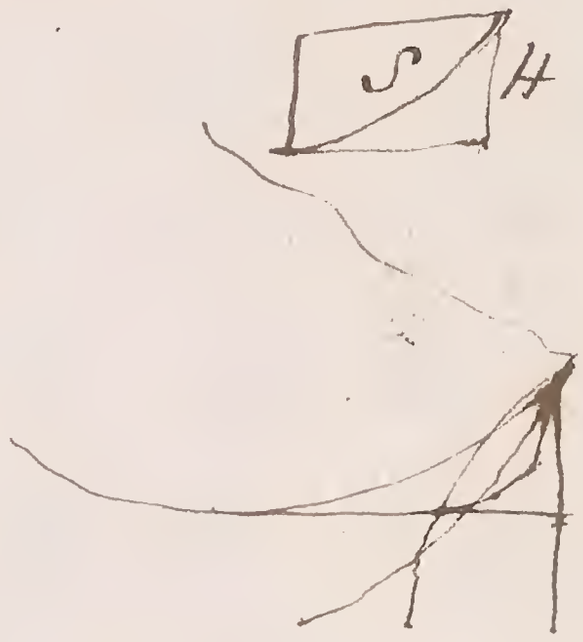


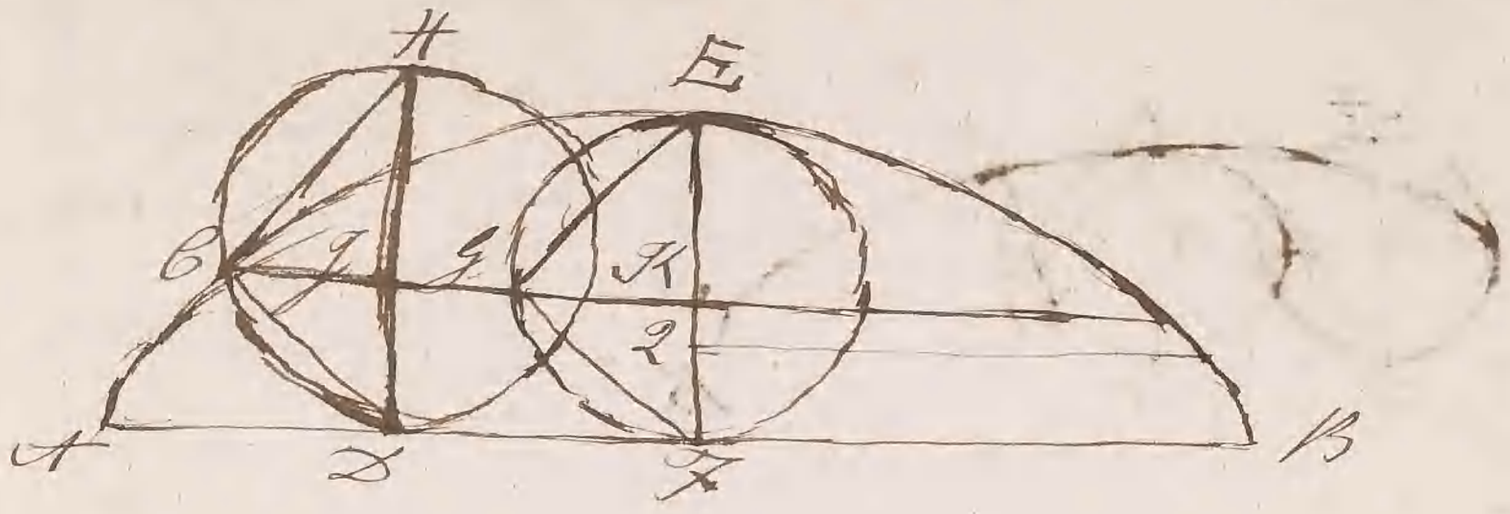
As a body will have acquired the same velocity in descending through A B, B C, C D &c seperately as through E A, F B, G C, H D it is evident that it will have acquired the same velocity in falling the A B, B C &c together as through all the perpendicular height, to E H. And if there be any number of planes of the same perpendicular height, a body will acquire the same velocity in falling through any of them, whatever be the lengths or inclinations of the planes.

If two similar and similarly situated series of planes or two similar and similarly situated arcs of a curve be the paths of the body the velocities and times of descent will be as the square roots of the lengths of the paths, of the heights or of any homologous sides belonging to the figure.

As the velocity acquired by a body descending through both of the series of planes or curves (which may be considered as the number of infinitely small contiguous planes) is as the square root of the heights, the velocity acquired in one will be to that in the other as the square roots of the heights, or, as the figures are similar & any homologous sides is.

When Universally $T : \frac{S}{V}$ $\therefore \frac{S}{\sqrt{H}} : \frac{S}{\sqrt{S}} : \sqrt{S}$





Of the Cycloid

Let the circle touch the right line A B in the point A; if the circle be rolled along the line, the curve described by the point in which it touched the line is called a Cycloid in one revolution of the circle.

The right line E D bisecting the base at right angles is called the axis and the point E the vertex; the circle G D H is called the generating circle.

If the generating circle be described about the axis and any right line as G K be drawn parallel to the base, the whole line G K will be equal to the arc g E together with its right sine g K, or, which is the same thing G G g is equal to g E.

Describe the generating circle so as to pass through the point G join G D & g F. It is evident from the definition that A D is equal to the circumference of the circle and consequently the half A F to the semicircle, and the part D F to the arc G D, D F is therefore equal to the arcs G H or g E but D F, G g are equal as they are parallel by construction and equal because the lines G D & g F are parallel, the arcs G D, g F and consequently the angles G D A & g F D being equal to one another, therefore

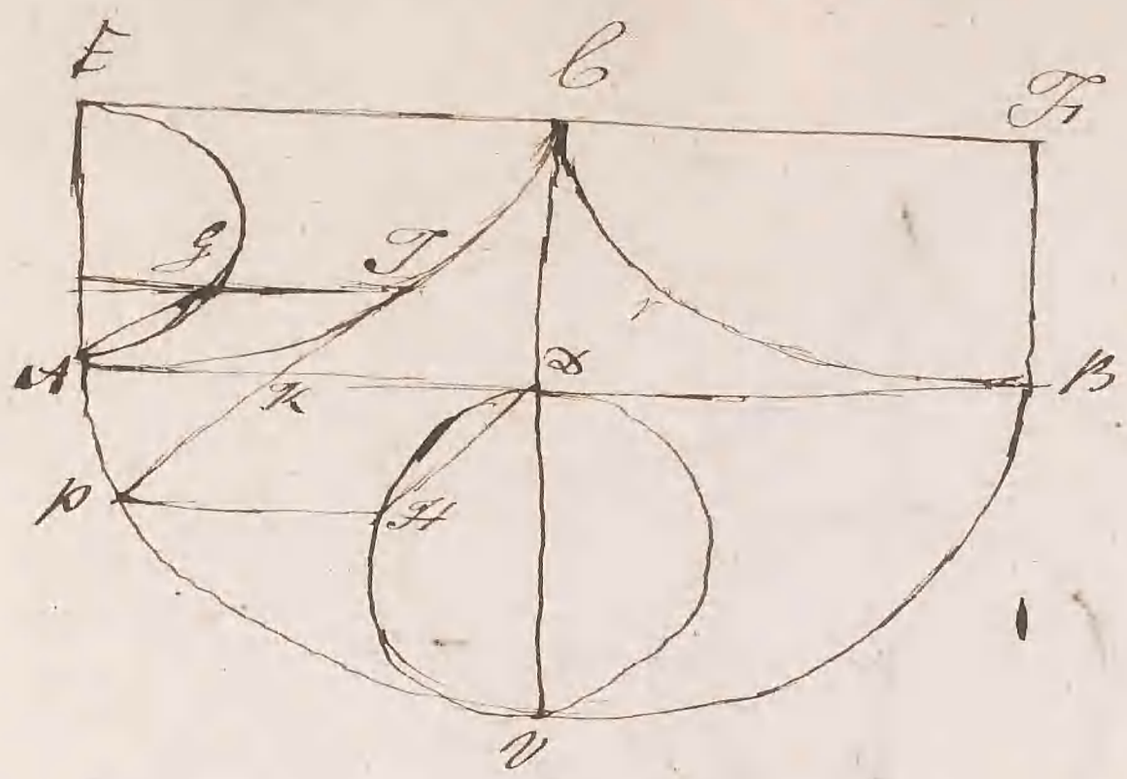
G g = g E or G K = G E + g K

Ste. The same construction remaining, the chord gE is parallel to the tangent to the curve at the point C.

Suppose the generating circle to move an infinitely small space then as AB is less than the curve of the cycloid the point D may be supposed at rest whilst C moves an infinitely small space, which will be an arc of a circle touching the curve at C and having a common tangent with it at that point which will evidently be parallel to Eg.

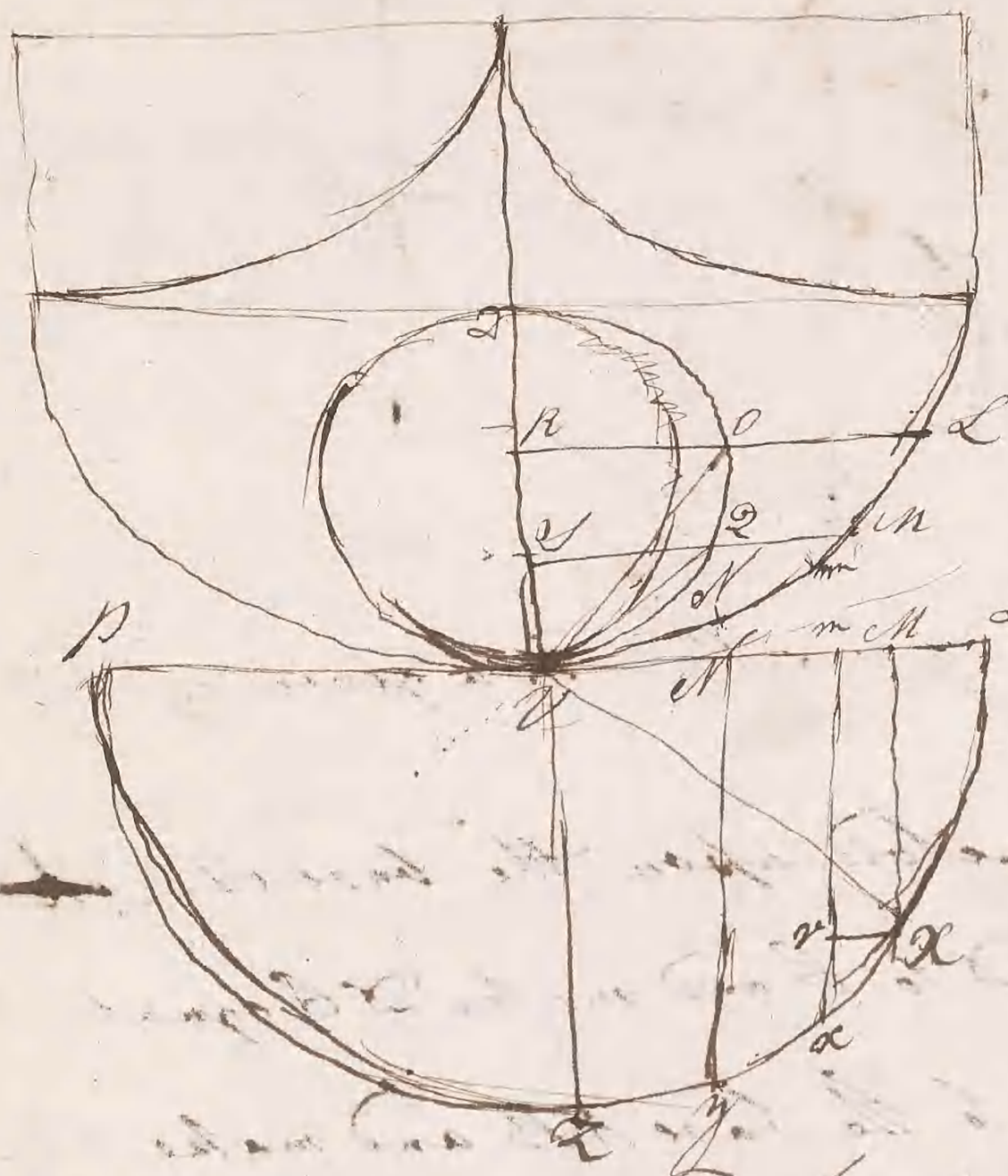
If through C any point L in the cycloid there be drawn a line parallel to the base meeting the circle about the axis in M; EL will be equal to twice EN.

Draw R very near to L, draw EN & MR touching the circle EgM in the points E & M, produce EN to P. Then as RM is an infinitely small space it may be considered as a part of the tangent MN, and the triangle RMP will be similar to ENM and consequently isosceles therefore a perpendicular drawn from the angle R to the base will bisect it, join ER, which being infinitely near to EO and the angle at O being a right-angle they will be equal to one another and MO will be the increment of the chord, as L is the increment of the curve may be considered as a part of a tangent at the point L, it will be parallel and equal to MP which is double of MO, therefore the increment of the curve is double that of the chord and as they begin from the same point the one will be always double of the other



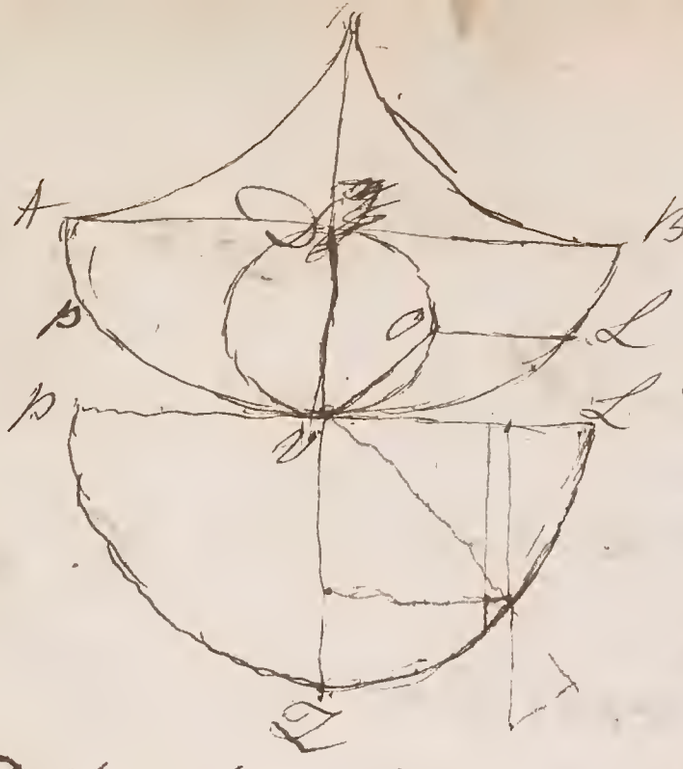
To make a body oscillate in a given cycloid.

Let A D B be a cycloid upon the base A B; ~~[illegible]~~ produce the axis D D to b and make D b equal to D D, through b draw E F parallel to the base A B and make b E, b F equal to A D & D B; Let two semicycloids A g E, B T F be placed upon the semibases E b & b F and let them touch A B in their vertices A & B. Now let a body be suspended at the extremity A of a string exactly coinciding with the semicycloid A D b & let it fall from A to B by its own gravity; the path of the body A is the given cycloid. About the axes E A & D V describe the generating semicircles E g A & D H V. Then it is evident that the lowest point through which the body A will pass is the vertex D. (the string being then in a situation perpendicular to the horizon.) Suppose the body to be in the point P and through P and D, which ~~[illegible]~~, draw P H & T g parallel to the base, join g A & D H; Then as P T P H & T g parallel to the base A B and make T g equal to it, P T is double of g or a tangent to the curve A T and equal to it, P T is double of g or T K and P K, K T are equal, therefore the points P & T or H & g are equidistant from the base A B, consequently the arcs H D & and the angles D H & D g A of equal circles, are equal to one another, therefore H D, or A K = are A g ∴ K P or P H = are g E or H V ∴ P is a point in the cycloid whose base is A B and axis D V, the same may be proved of any other point in the path of A



If the axis DV of the a cycloid be perpendicular to the horizon and its vertex V the lowest point, & a body begin to oscillate towards V from any point L in the cycloid; its velocity any point M will be as $\sqrt{VL^2 - VM^2}$; or, if a circle be described with a radius VL equal to the curve, as the right sine or ordinate drawn through the corresponding point M. Through L & M draw lines parallel to the base meeting the generating circle in O & Q and the axis in R & S, join V O & V Q; then as the velocity acquired at the point M is equal to that of a body falling perpendicularly from R to S which is as the square root of RS or of RV - SV or of QV × VD - VN × VD or of $VQ^2 - VN^2$ or of $VL^2 - VM^2$ or lastly $\sqrt{VX^2 - VM^2}$ or MX.

The same construction being supposed; let a body uniformly move in the circle LQP with a velocity equal to that acquired at the vertex V of the cycloid, it will describe any arc of the circle XY in the same time that a body beginning to oscillate at L describes a corresponding arc of the cycloid LMY. Draw mx very near to MX and draw xr at right angles to it. As Mm is an infinitely small space we may consider it as described with the uniform velocity MX, and Xx is described with the velocity VX, then as the triangles Xxr & VXM are similar Xx : xr or Mm :: VX or VZ : MX, that is the spaces are as the velocities with which they are described & the times are equal; and as Xxy and MmY are made up of parts Xx & Mm, the whole XY will be described in the same time as MmY. And the arcs of the circle will be proportional to their



In a given cycloid the time of any oscillation is to the time of descent through the axis, as the periphery of a circle to its diameter.

The time in which the semicircle LZP is described uniformly with the velocity acquired at V is to the time in which L V is described with the same velocity :: as the periphery of a circle to its diameter; but the time in which the semicircle is described is equal to that of a compleate oscillation, and the radius VL, or the arc VL, or twice the chord OV is described in the same time that a body descends to acquire that velocity that is along the chord OV — the time of falling down the axis DV; therefore the time of any oscillation is to the time of descent through the axis of a given Cycloid as the Periphery of a circle to its diameter.

From this it is evident that in a given cycloid (the axis, and consequently the time of descent through it, being given) the all oscillations are performed in equal times.

Cor: 1 If two bodies be suspended by two equal strings, they will perform their oscillations in equal times; therefore if they descend perpendicularly they will be equally accelerated and their weights will be as their quantities of matter.

The space described by a body falling from rest in the time $\frac{1}{2}$ is to half the length of the string by which if the same body were suspended it would have performed a compleate oscillation $\frac{2}{2}$ :: as the periphery of a circle to its diameter $\frac{2}{1}$ (or let the body fall through half the length of the string; then the time of descent will be to that of oscillation as D to P; Now let the time of descent be increased so as to be equal to the time of oscillation and the space described will be increased in the duplicate ratio

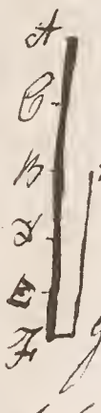
$\frac{1}{2}$ $\frac{1}{2}$ :: String $\frac{1}{2}$ :: P R :: D R

The time of oscillation in a cycloid, when the length of the str
and the force of gravity are both variable, is as the square roo
of the length directly and of the Force inversely. The
time of oscillation is as the time of descent through the a
which time is as the square root of the axis or of twice the
axis or of the length of the string directly and of the Fo
inversely; for if the force be given the space is as the square o
the time and if the time be given as the velocity and consequently as
force; if neither be given, the Space will be as the force a
the square of the time. Therefore the time will
be as $\sqrt{\frac{L\,string}{F}}$ or the time in the subduplicate ratio of the
Length of the string directly and of the Force inversely
Cor 2. Length of the string

Cor. 2d of the former. Prop. Hence the space descended freely in any time
by the force of gravity may be found.
For let s be the length of a pendulum oscillating in that time
then the Time of descent down $\frac{s}{2}$ is to the time of oscillation
as D to P. But the spaces are as the squares of the
times therefore $\frac{s}{2} : x :: D^2 : P^2$
in a second $\frac{39.1}{2} : x :: 1 . g . x$

To prevent the expansion or contraction of pendulums by a Gridiron

The expansion of brass is generally allowed to be to that of steel as 5 to 3. If steel expands $\frac{3}{10}$ of an inch brass of the same length expands $\frac{5}{10}$; therefore the expansion of a bar of steel is exactly equal to one of brass $\frac{3}{5}$ of its length.

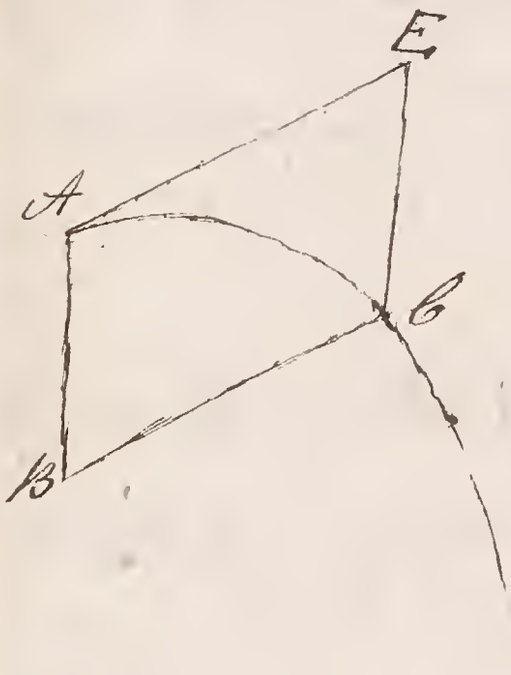


Let AB be the length of any pendulum to which the gridiron is to be applied; bisect AB in C & make BD, DE, EF each equal to AC, then AF will be to FB or GH as 5 to 3; therefore if AB be made of steel and GH of brass and AB and GH be made fast at F, and the whole suspended at A, GH will expand upwards the same as AF does downwards and the point B (the centre of oscillation) will remain always at the same distance from the point of suspension whatever be the change of heat or cold. And since the effects will be the same, however the brass and steel is disposed provided that the steel expands downwards and the brass upwards, they may be placed (as in this figure) where the black lines represent the steel and the fine ones the brass.

In theory only five bars are necessary two of brass and three of steel; but in practice it is necessary to have nine as in the figure K in order that the bar on which the pendulum is suspended may be equally disposed on both sides to prevent such tremulous motion or bending of the bars as would otherwise take place.

Of the motion of Projectiles.

The curve described by the motion of a body, thrown in any ~~line~~ direction, not perpendicular to the horizon, in a non-resisting medium is a Parabola: And the velocity in any ~~part~~ point of that parabola is equal to that which a heavy body would have acquired by falling perpendicularly through a fourth part of ~~the~~ parameter of that diameter in whose vertex the body then is.



Let a body be projected from A in the direction AE by a single impulse which would carry it through AE in a certain time, & let AB be the space described by a body falling freely by the force of gravity in the same time, compleat the parallelogram EiABC and it is evident that if the body, acted upon by these two forces at the same time it will be found, at the end of the given time, in the point C, but its path AC will be a curve, as AE is a single impulse and the force in AB constant, as AB is the time in which it is described, AE^2 or BC^2 is as the square of that time, but AB is as the square of the same time, therefore AE^2 or BC^2 is as AB; and C is a point in a parabola one of whose diameters is AB, BC an ordinate to that diameter and AE, parallel to BC, a tangent to the curve in the vertex A; in the same manner it may be proved that the point C will be in a parabola at the end of any other time, therefore the locus of all the points C is a parabola, the parameter of whose diameter AB is $\frac{BC^2}{AB}$ or $\frac{AE^2}{EC}$.

which was the first part to be proved.

Q) The ~~projectile~~ velocity ~~force~~ on any point of the parabola is equal to that which a body would acquire in falling through a fourth part of the parameter of ~~which has that point for the~~ vertex of its diameter.

The ~~projectile~~ velocity at any point A is that with which a body would describe the space AE, in the same time that it would fall from E to C, but the velocity acquired at C is to ~~that~~ velocity, with which a body will describe AE in the same time, as EC to $\frac{AE}{2}$; the same velocity acquired by falling from E to C it is to that acquired in falling through a fourth part of the parame as ~~E to~~ $\left(\frac{AE^2}{4EC}\right)$ as $\sqrt{EC}$ to $\sqrt{\frac{AE^2}{4EC}}$ or as EC to $\frac{AE}{2}$ therefore as the velo acquired at C has the same ratio to the velocity at A. as it has to that acquired by falling through a fourth part of the parameter th velocities are equal to one another. Q E D.

Cor 1. If a body be projected in the direction AE, and that the param of the vertex A of the parabola to be described be equal to $\frac{AE^2}{EC}$; the parabola will pass through the point C.

Cor 2 In the same or different Parabolas the parameters are to to one another as the ~~projectile~~ squares of the velocities at their vert

Cor. 3. Whatever be the parabola to be described if the velocity is given the parameter will be given a

Cor 4. Hence if a body is turn'd up perpendicularly to horizon with the velocity it has in any point of a given Parabola it will always rise to the Direc

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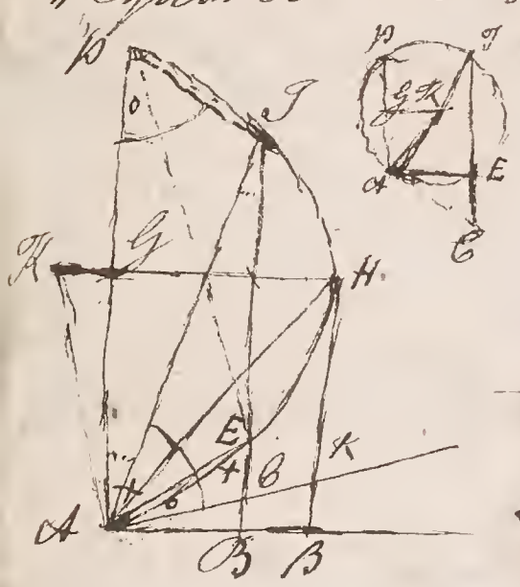
Cor. 5. (to the problem.) The greatest time in a direction AP perpendicular to the horizon, is that in which a body would fall freely through P A the parameter. As the body in this case rises and falls perpendicular through a fourth part of the parameter, it is evident that in the same time it would have fallen through the whole parameter or four time that space.

Problem

Given the velocity with which a body is to be projected from a given position
must be thrown to
required the directions in which it ~~will~~ hit a given mark

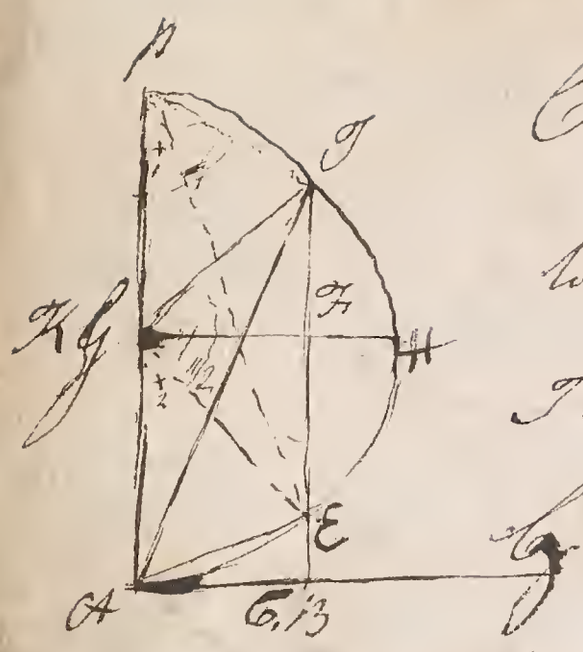


As the velocity is ~~given~~ given the parameter of the parabola will be given; as it will evidently be four times the space through which a body ~~must~~ fall to acquire the given velocity. Let C be the mark in the given right line AC; through the point A draw AP perpendicular to the horizon and equal to the parameter found, let KH bisecting AP at right angles meet KA perpendicular to AC drawn through A in. K, and the circle in H through C draw BCEI meeting the horizontal line AB in B and the circle, if possible, in E & I, join AE, AI these are the directions required. For, as the triangles APE, EAC and APF, AIC are similar AE^2 is equal to PA × EC and AI^2 to PA × IC or $\frac{AE^2}{EC}$ and $\frac{AI^2}{IC}$ both equal to AP the parameter, the parabola will therefore pass through the point C, (by Cor: 1st to the both Proposition) QEF

Cor 1st. It evident that AH bisects the angle CAP the measure of the visible distance between the mark and the Zenith.

Cor 2. When the Points E & I or the directions AE & AI coincide, the distance of the mark is the greatest; or a body will fly to the greatest distance with a given velocity when projected in that direction. For CI then becomes a tangent to the circle and the points E & I coincide at H.

Cor 3. Any two directions AI, AE in which the same mark may be hit, are equidistant from the direction AH that is the angles IAH & EAH are equal. For they are upon equal arcs in the same circle.



Cor 5th. The distance of the mark, A B or A B, in a horizontal line, is as the sine of twice the angle of elevation E A B or I A B. For if B I meet G H in F, it is evident that A B or G F is equal to the sine of the angle E G O or of I G O that is to the sine of twice the angle of elevation.

Cor 6th. The greatest distance of the mark upon a horizontal line is half the parameter, or radius, the greatest possible sine: consequently the angle of elevation must be half a right angle to carry the body to the greatest possible distance with a given velocity.

Cor: 6th. If a body be projected in the direction A E, its greatest perpendicular altitude is equal to one fourth of E B. For, bisect A B in D and draw D K at right angles to it, then D K will be the axis of the parabola A B an ordinate and A E a tangent at the extremity of the ordinate, D K will therefore be bisected in V the principal vertex and D V equal to ½ D K or ¼ E B; but E B is the cosine of the arc A E or of twice the angle of elevation, therefore the greatest perpendicular altitude is as the cosine of twice the angle of elevation

Cor 7th. The greatest altitude of a project thrown in a direction A P, is equal to ¼ of the parameter, and is therefore given. For in this case A E and E B coincide with the direction A P B and the altitude D V or ¼ B E becomes now ¼ A P.

Cor 8th. The time in which a project move from A to B is as the chord of twice the angle of direction or as the sine of the angle of elevation. For body moves from A to B. and A E is as the time (as was proved before) and ½ A E = sine of elevation.

Optics

A Ray of Light is supposed to consist of minute particles of matter following one another in a straight line, & which they will continue to move if not interrupted.

Let a ray in the direction AC impinge on the plane NS; the angle ACP is called the angle of incidence; if the ray is refracted CEQ is called the angle of refraction if it be reflected PCB is called the angle of reflection;

Laws of Reflection and Refraction.

1st The angles of reflection and refraction are in the same plane with the angle of incidence.

2 The angles of incidence and Reflection are equal.

3 Refraction is made towards the perpendicular, into a denser medium into a rarer medium from it

4th The sines of incidence and refraction are to one another in a given ratio.

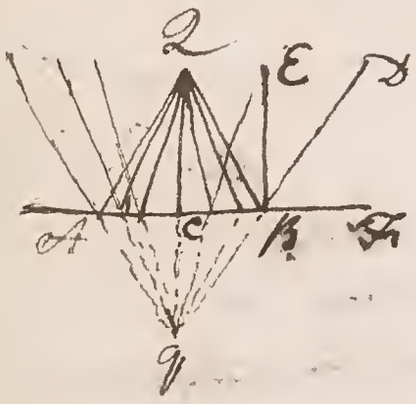
It is evident that if a ray falls perpendicularly upon any plane it will be reflected or refracted in that same direction the angle of incidence being nothing.

If a ray falls in any direction upon a convex or concave spherical surface its reflections and refractions are to be estimated as if the ray impinged upon a plane touching the sphere at the point of incidence; as the perpendicular is common to both the sphere and the plane, and alone determines the motion of the ray.

A Focus is that point from which rays are diverging or to which they are converging; if the rays are parallel the focus is supposed to be removed to an infinite distance. Any number of these adjacent rays is called a pencil.

If a pencil of parallel rays falls upon a plane surface it will be reflected or refracted parallel; because all the angles of incidence, and consequently all the angles of reflection or refraction, are equal to one another.

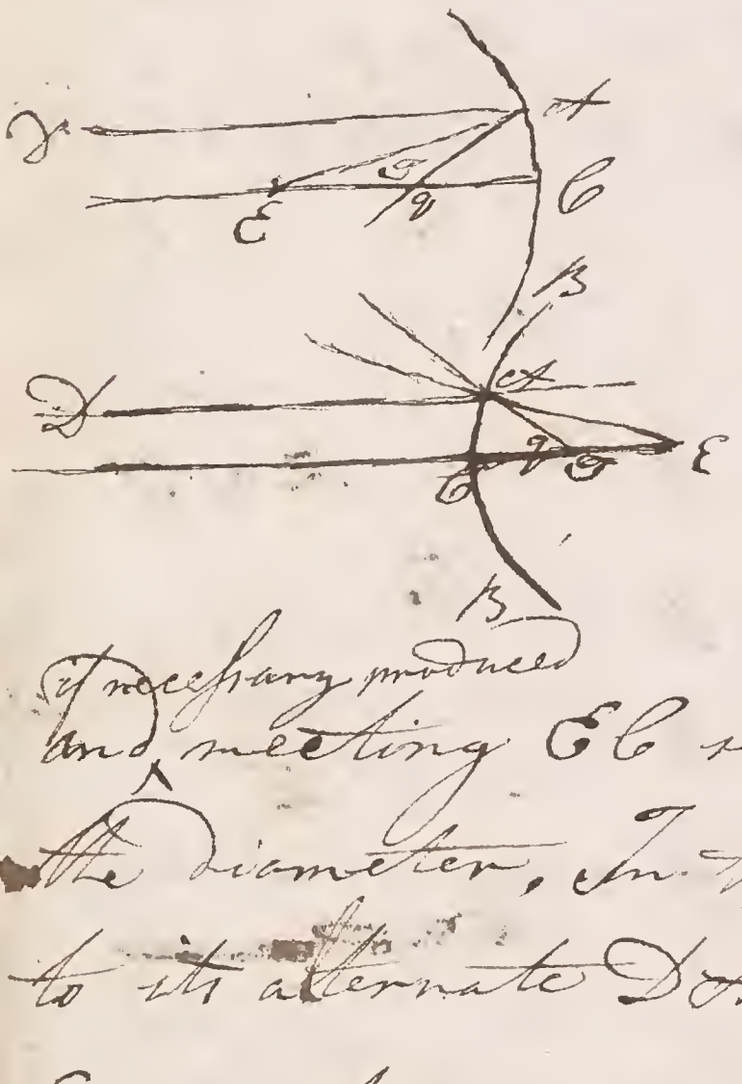
If a pencil of converging or diverging rays falls upon a plane surface the foci of the incident and reflected rays are on opposites sides of the reflecting plane and at the same perpendicular distance from it. Let Q be the focus of a pencil of rays reflected from the plane surface AB; let Qc be that ray which is perpendicular to AB, produce it to q and make cq equal to cQ. Then q will be the focus of the reflected rays. Let QB be any ray, join qB and produce qB to D, draw CB at right angles. Then as Qc & qc are equal, the angles at c right ones and cB common QBc is equal to (cBq or) DBA and QBC equal CBD, therefore BD, the reflected ray, produced passes through q; the same may be proved of any other ray; therefore q is the focus, i.e. that if the incident rays converge towards q they will be reflected to Q.



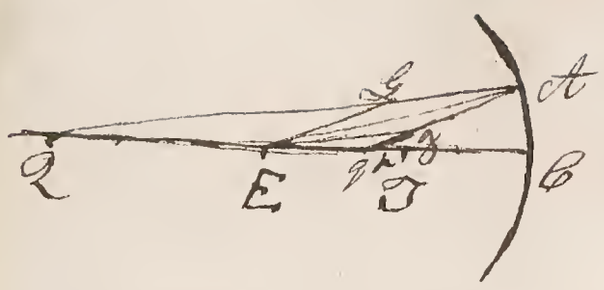
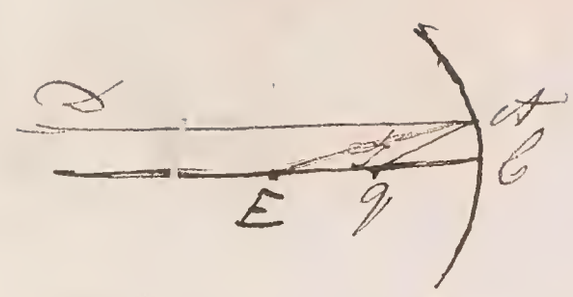
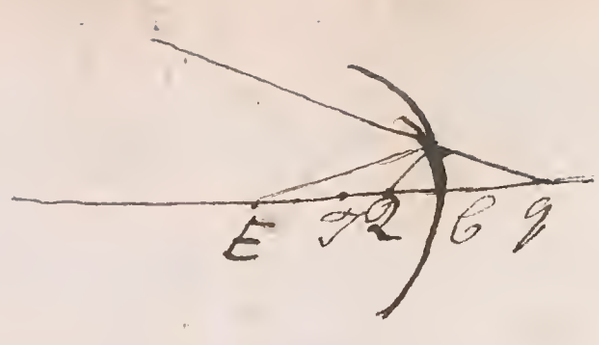
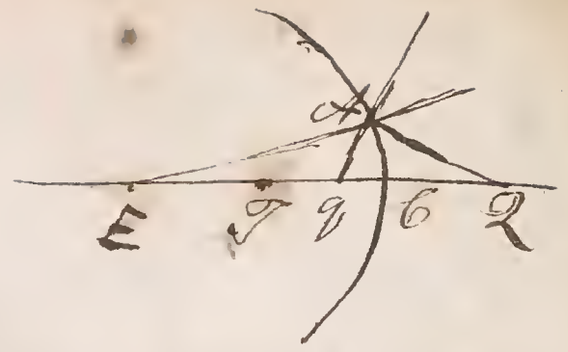
The refractions of a single Ray.

If a ray falls upon a refracting medium terminated both ways by two parallel plane surfaces it will emerge in a direction parallel to the first as the angles of deviation are equal at its entering and leaving the medium. The same thing is nearly true if the medium is terminated by two concentric arcs of a sphere as the lines drawn touching will be nearly parallel.

If a pencil of parallel rays falls upon a spherical reflecting surface the focus of after reflection of those rays which fall near perpendicular to the surface is found in the diameter parallel to those rays and halfway between the center and surface. Let AB be any reflecting spherical surface, let EB & DA be two of the parallel rays, one of which, EB passes through the center bisect EB in D; join DA; then DA will be reflected in a line Aq making the angles at A with EA equal



if necessary produced and meeting EB somewhere in q. That point q will bisect the diameter, In the triangle Eq the angle at E (which is equal to its alternate DAE) is equal to the angle at A, Hence the Eq & qA are equal; but when DA is very near to EB the triangle vanishes and Eq becomes equal to half EA or EB; Therefore q coincides with D.

Let Q be the focus of a pencil of rays incident upon a reflecting sphere; let Qb be one of those rays which passes through the center and let I be the point of bisection as before; take a point g, on the same side of I with Q, so that gI, : IE :: IE : Ig are proportionals. g is the focus of the reflected rays.

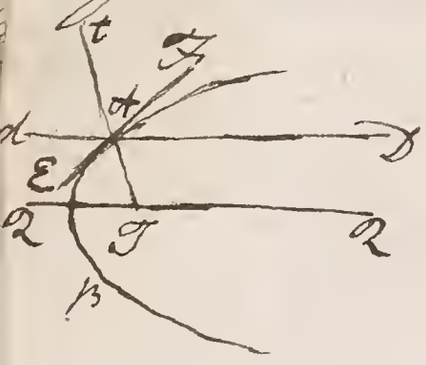
Let AQ A be another incident ray falling nearly at right angles to the sphere; join E A and let A g be the direction of the reflected ray meeting Eb in g; through E draw EG & Eg parallel to Ag & AQ then as the triangles QGE & EGg are similar QG : GE :: EG : Gg but is (because when QA is very near Qb G & g coincide with I) QI : IE :: IE : Ig therefore g & g are the same points; and g is evidently the focus as it is the intersection of two of the reflected rays Ag & bg.

From these propositions ~~it is evident~~ this table is deduced

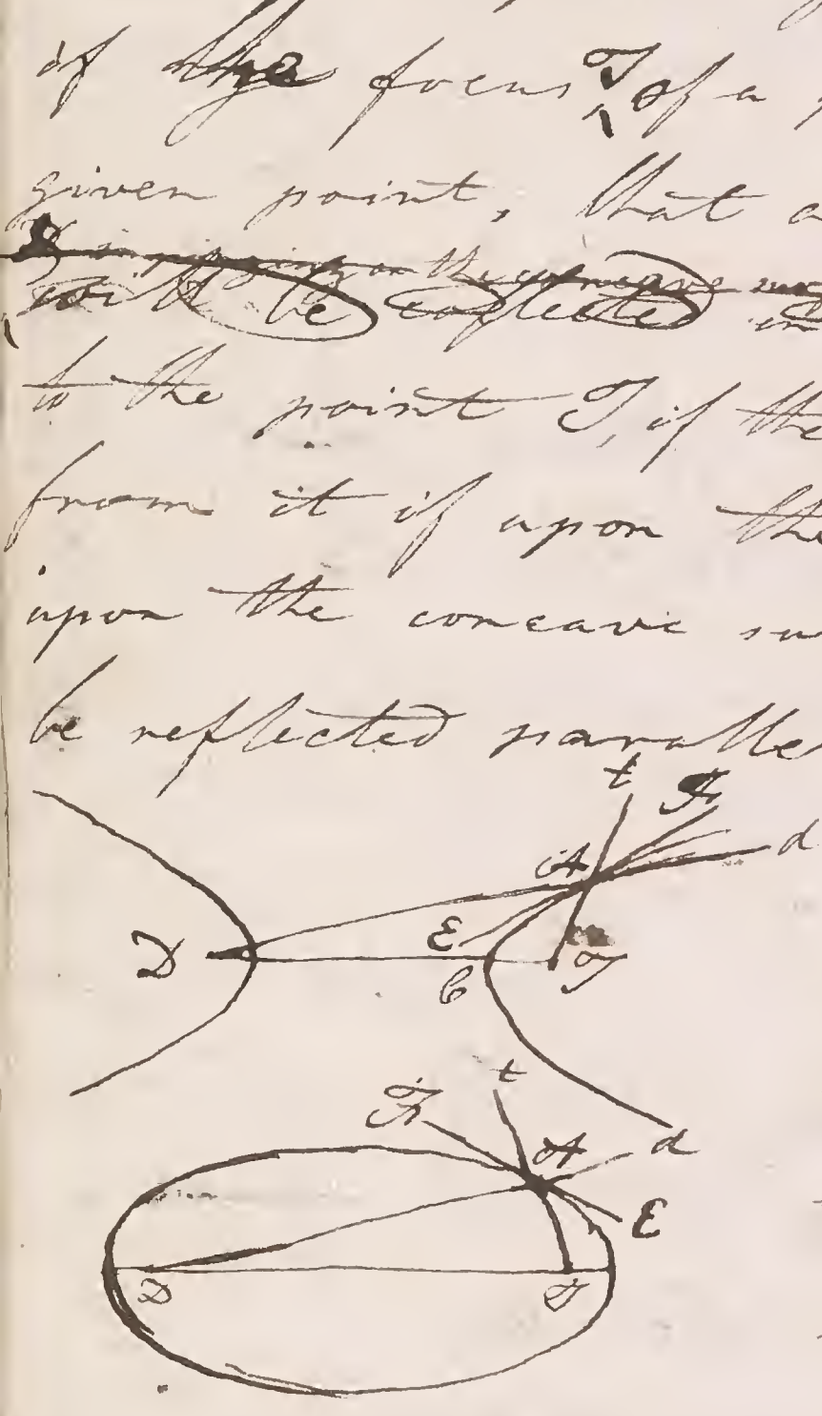
Q at an infinite distance to the left of E (i.e rays parallel.	g at I
Q at a finite distance to the left of E	g between E & I
Q at E	g at E.
Q between E & I	g at a finite distance to the left of E
Q at I	g at an infinite distance to the left of E
Q between I & b	g to the right of b (i.e. reflected diverging.
Q at b	g at b.
Q to the right of b. (i.e. ~~parallel~~)	g between I & b.
Q at an infinite distance to the right of b	g at I.

The first proposition, of the focus of rays reflected from a plane, may be brought to this. For, as Ig : Ib : : IQ, (IB - Ig or) gb : (IQ - Ib or) Qb : : Ig : IB. but in this case E & I are supposed at an infinite distance, : Ig & which are infinite : : with a finite difference may be called equal; therefore gb = bQ.

A pencil of rays may be made to converge to or to diverge from any given point by the help of reflecting surfaces generated by the motion of some conic section round its axis.



Let AB be a parabola, T its focus 2 T 2 its axis produced A D any diameter meeting the ~~curve~~ surface in A and produced to d, E A F a tangent at the point A & D A a line joining the focus and A, produced to t; the tangent E A bisects the equal angles t A D & D A T. From this it is evident that if the focus T of a parabotoid, described by A B, be placed in a given point, that a pencil of rays parallel to the diameters ~~of the surface~~ will be reflected to the point T, if the rays impinge upon the concave surface, and from it if upon the convex; rays diverging from T and impinging upon the concave surface or converging to upon the convex will be reflected parallel.



Let D and T be the foci of an ellipse or opposite Hyperbolas, let A be any point in the section from which A D & A T are drawn to the foci and produced to d & E, let T A E be a tangent at the point A; T E bisects the equal angles t A D, d A T. From this is evident if the focus T of a reflecting spheroid or hyperboloid is placed on the given point that rays diverging from in the one focus and converging to the other the hyperboloid, and impinging upon the concave surface will be reflected converging to the other, in the convex if they converge to one focus in the spheroid or diverge from it in the hyperboloid.

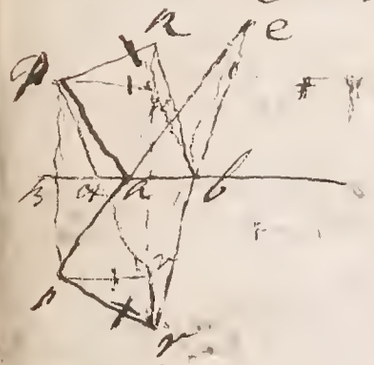
~~An object is supposed to be made up of a number of points rays diverge in every possible direction~~

An object is made up of a number of points from each of which rays diverge in every possible direction and an image of the respective foci of those rays after reflection or refraction.

Images formed by the reflection of a plane surface are equal and similar to the objects; and are at the same distance from the back of the plane that the objects are from the face of the plane, and have the same situation with respect to it that the objects have with the face.

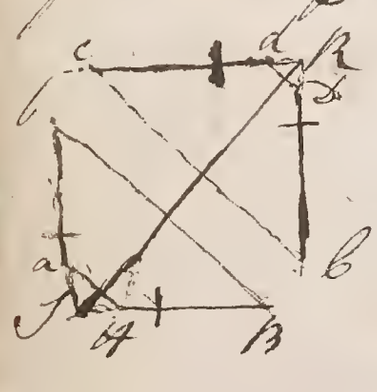
It is evident from the definition that every point in the image is at the same distance from the plane that with the corresponding point in the object, therefore the whole image will be equal to the whole object and will have a similar situation to the back that the object has to the face of the plane.

The distance of the object from the eye is equal to the incident or reflected rays taken together. That is

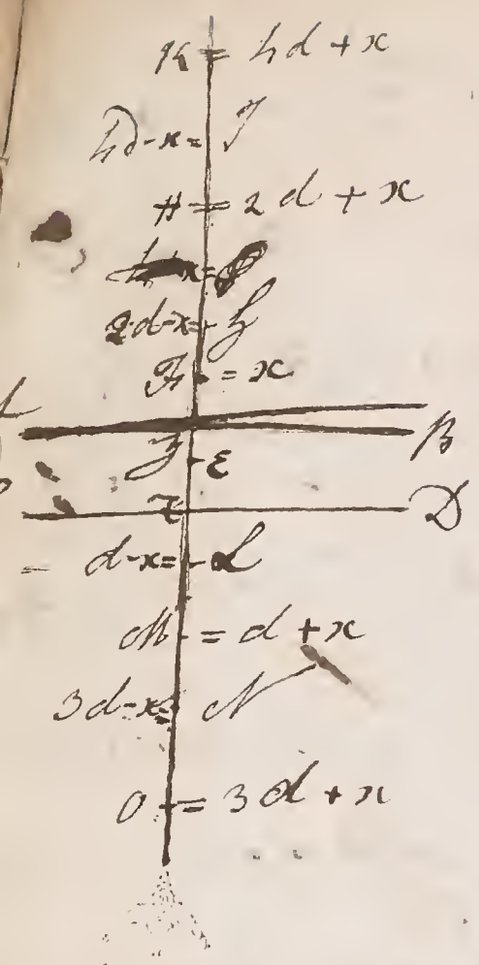


upon er, the distance of the eye at e from the object or image pr, is equal to ea + aP or eb + bR.

If from any point in the object PR, parallel to the plane be drawn the lines Pp, Rr, the part AB of the plane between those lines will be equal to half the object: for the triangles PBA & PBp are similar PB, BP equal. From this it follows that if an eye be placed in the plane of an object parallel to the reflecting plane, it will see the image through a part of the plane whose length and breadth is half that of that of the object, that is, whose area is a fourth part of the area of the object.



If a plane speculum RS be inclined to the horizon in an angle of 45 degrees with its face downwards; an upright object CD will have its image cd in a horizontal position and an image a b of an object AB lying horizontally will be vertical; because the angles CRD & cRd are both equal to half a right

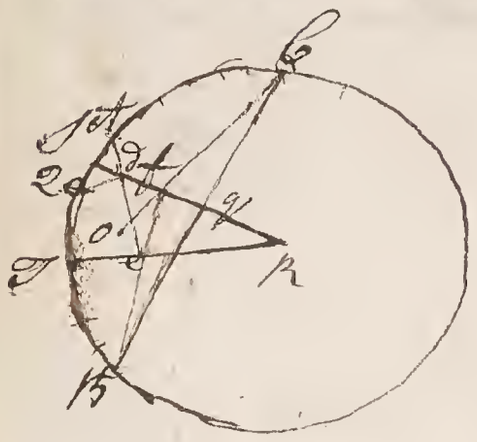


If an object be placed between two specula the rays diverging from it will be reflected backwards and forwards from them so as to form an infinite number of images in each speculum.

Let E be the object and AB, & D the planes, let x be equal to GE and D to yZ, now to consider those reflections only which begin with AB, if yF be taken equal to x it is evident that F will be the place of the image in the speculum AB and this being considered as an object to the speculum D, the distance of its image M from D will be equal to ZF or d+x, this being again considered as an object the distance of its image from AB will be equal to MG or 2d+x and if ZO be taken equal to Zy or to 3d+x O will be the place of the image and so on; in the same manner it may be proved that L, J, N, I &c will be at the distances x-x, 2d-x, 3d-x, 4d-x from the speculum D; therefore if D be multiplied into each even number and x added to the product and subtracted from it, the sums and differences will express the distances from the speculum AB and if x be added to and subtracted from D multiplied into the odd numbers will express the distances from D.

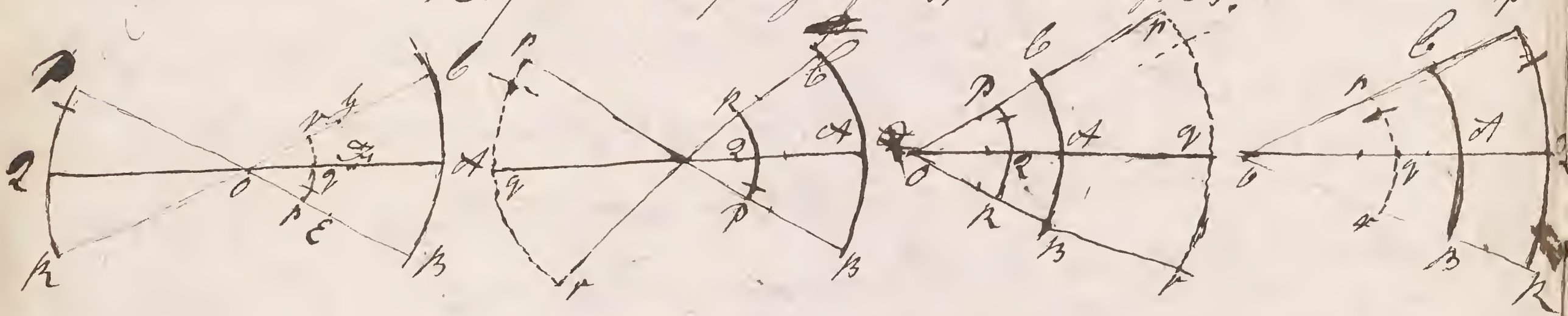
If an object is placed between two specula (K, K) inclined to one another and meeting in an angle (at K) several images of it will be seen by an eye placed any where between the specula standing in a circle whose center is K and which passes through the object and if the angle DKD is an aliquot part of four right angles the number of figures existing in the object itself will be equal to the number of parts. From the center K describe a circle passing through the object, draw DK at right angles to the plane AK, then it is evident that the sector AKB will be the image of DKD and if AB be drawn at right angles to DK, DK will be the image of DK and DK & of DK &c but the object is in the arc DD therefore it will be

seen as often as there are angles in four right ones equ
to 1 R 2 or in the arcs in the whole circle equal to 2 &



a at 2 be an object between the speculums 1 R, 3 R as
before, and 6 one of the images, the distance of 06
from the eye at o to the image at 6 will be equ
to 2 d, d e, e f, & f 6 the lines described
by the rays in passing from the object to the eye. Because th
triangles C f g & B f g are equal and similar f 6 & f B are equal
for the same reason e B & e a t & d a t & d Q are equal therefore
o 6 = o f + f e + e d + d Q.

Reflection of images from spherical surfaces.



First suppose the image P Q R to be an arc concentric with
the speculum; from any number of points P, Q, R draw lines
passing through the center and meeting the speculum in B, A, C &c,
viz. o B, o a, o C in E, F, G &c; take the points p, q, r
so that p E : E o : E P. F q : F o : F Q. G r : G o : G R the figure
p q r made up of those points which are the foci of the rays
diverging from P, Q, R &c (as has been proved) will be the image
of P Q R in the speculum C A B and it (p q r) will be concentric
with the object or speculum.

Refraction.

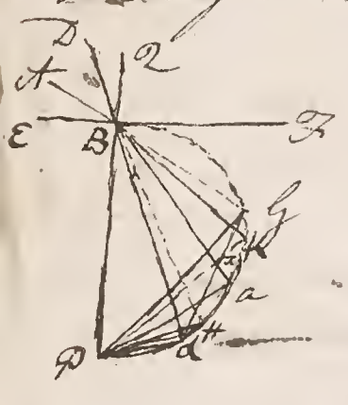
1st The angles of incidence and refraction are in the same plane,

2 If the refracted ray be turned directly back to the point of incidence, it will be refracted into the same line before described by the incident ray.

3 Refraction out of a rare medium into a dense is made towards the perpendicular, that is, the angle of refraction is less than the angle of incidence. And the converse.

4th The sines of incidence is in a given ratio to the sine of refraction

The angles of deviation, (that is, the angle between the refracted ray and the incident ray produced) ~~varies~~ increases with the angle of incidence; though not in the same ratio.

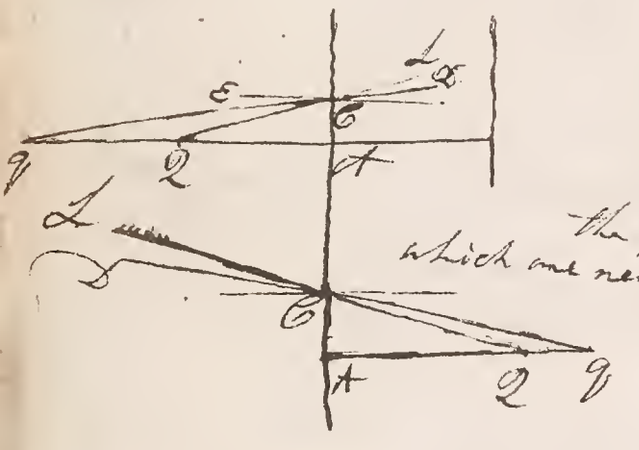


Let DB & AB be two rays incident upon the plane EBF at the point B, let QB be the perpendicular at that point produced to P, about any line BP as a diameter, describe a semicircle meeting the incident rays produced in G & H and the refracted ones (Ba & Bd) in a & d. G Ba is greater than H B d. Take the arc GK (on the side next to Ba) equal to the arc H d, (or the angle H B d,) draw K B & K P & and join G d which must meet P K in some point c. As the points G, a, H & d are in a semicircle, the lines PG, Pa & PH, Pd are the sines of incidence and refraction of the rays AB & DB, therefore (from the 4th law of refraction) G P is to Pa as H P to Pd or (because the triangles H Pd & G Px are similar) as G P to Px therefore Px & Pa are equal and P K is greater than Pd and less than G P by construction; P K must therefore, cut the angle G B a or the arc G a and consequently the angle G B a or the arc G a is greater than the arc (G K or) H d or the angle G B a than the angle H B d.

A pencil of parallel rays falling upon a refracting plane surface will be refracted parallel; as all the angles of incidence will be equal

4

If a pencil of converging or diverging rays falls upon a plane refracting surface, the focus of the refracted rays, very near the perpendicular is found on a line drawn from the focus of incidence at right angles to the plain and at a distance from (the plain) which is to the distance of the focus of incidence as the sine of incidence to the sine of refraction



Let Q be the focus of a pencil of rays incident upon the refracting plane Ct; the focus q of the rays after refraction will be found in the line Qt, and Qt will be to qt as the sine of refraction to the sine of incidence. Let Qb be one of the incident and qb one of the refracted rays, let Eb be the perpendicular at the point b; in the triangle qbQ, Qb is to qb as sine angle qtb to sine angle Qtb; but b is supposed to be very near t, therefore qb & Qb nearly equal to qt & Qt and the angles bQt & bqt are equal to the angles of incidence and refraction; therefore Qt is to qt as the sine of refraction to the sine of incidence.

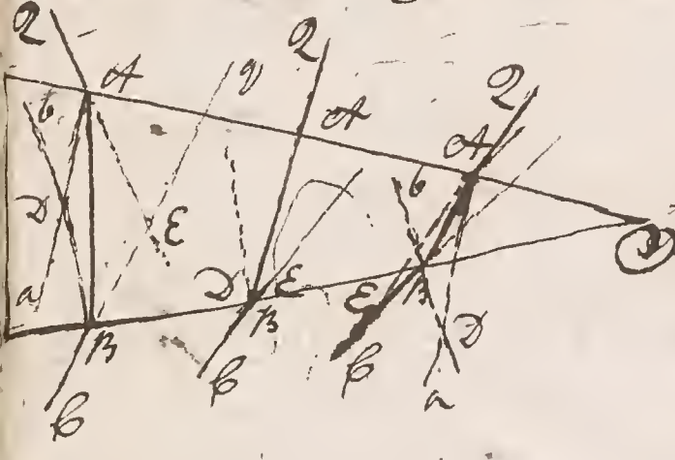
It is evident that if refraction is made out of a rare medium into a dense that the focus of refraction will be at a greater distance from the surface than the focus of incidence as the angle of incidence is in this case greater than the angle of refraction. And the converse

As tQ : tq :: R : I. (tQ as tq) or Qq will be to tQ as R is to R. For this reason an object placed under water appears raised above its true place one fourth of the depth; for the sine of incidence is to the sine of refraction out of water into air as 3 to 4 and Qq is to Qt (as R - I that is 4 - 3 or) 1 to (3 or) 4. If the object be parallel to the surface of the water it will appear raised one fourth of the depth, though still parallel; if it be not parallel the object and image will meet, if produced, at the surface for the object and surface must meet somewhere, and at every point the object is raised one fourth of the whole depth which at the intersection is nothing. For this reason a stick when it is partly put under water appears bent at the surface.

If a ray passes through a medium terminated ~~both ways~~ by two parallel planes its direction after passing through will be parallel to that it had before

Let 2 A be the incident and B b the emergent ray; then as the sides of the plane are parallel to one another the ray within the medium will be equally inclined to both

Therefore the angle of deviation will be the same at both surfaces, that is, the angle 2 A a B will be equal to the angle C b B g and consequently the rays 2 A & b B will be parallel.



If a ray 2 A B b passes through a medium terminated by two plane sides R S, T S inclined to one another in an angle R S T the emergent ray B b will incline from the incident ray 2 A towards the thickest part of the medium; and if the angle R S T and the angles of incidence and refraction are small as to be proportional to their sines, the whole angle of deviation 2 E g between the incident and emergent rays produced will be to the angle R S T as the difference between the sines of incidence and refraction is to the least of them, that is, 2 E g is to R S T as m − n : n.

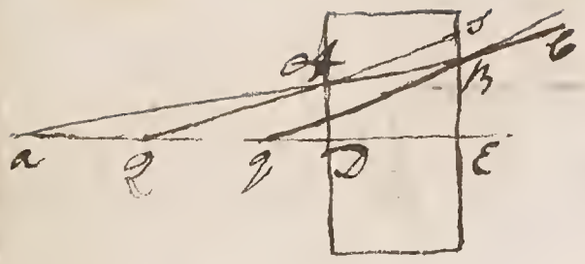
Let a A, B b be perpendiculars to the sides R S & T S and let them meet in D. make m : n :: sine ∠ a A E : sine ∠ a A B, and n : m :: sine ∠ b B A : sine ∠ b B E; it is evident that B b will be the emergent ray.

1st It is evident that B b produced will fall between A & S.

That is B b will incline to the thickest part.

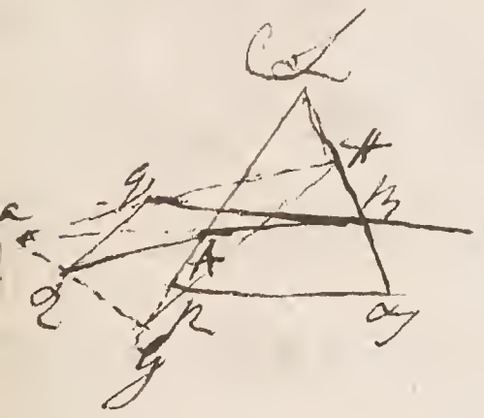
2 ly ∠ b B A : b B g :: n : m :: a A B : a A E. b B A : A B g = b B g − A B b) :: n : m − n :: a A B : B A E (= a A E − a A B)

Therefore (b B A ± B A a = b x A) R S T : (A B g ± B A E =) A E g :: n : m − n. Q E D.



If 2 be the focus of a pencil of rays diverging upon the medium a E, the focus q of the emergent rays will be found between 2 and the medium and 2 q the distance between the foci will be to D E the thickness of the medium as m - n to m. Let e be the focus of the refracted rays. Then a D : 2 D :: m : n :: a E : q E. Therefore a 2 : a D :: m - n : m :: a q : a E and 2 q : D E :: m - n : m.

Having the focus of incident rays given upon a medium terminated by two plane sides inclined to one another in a given angle to find the focus of the emergent rays



Let S K T be the medium terminated by the plane sides S T & S K meeting in an angle at S, let 2 be the focus of rays incident upon the side S K and let 2 A be one of the rays; draw 2 q at right angles to S K, produced if necessary and take q g to 2 q as m to n, then g will be the focus of the refracted rays; from A draw a H at right angles to S T and take g H to a H as n to m, then g will be the focus of the emergent rays, as is evident.

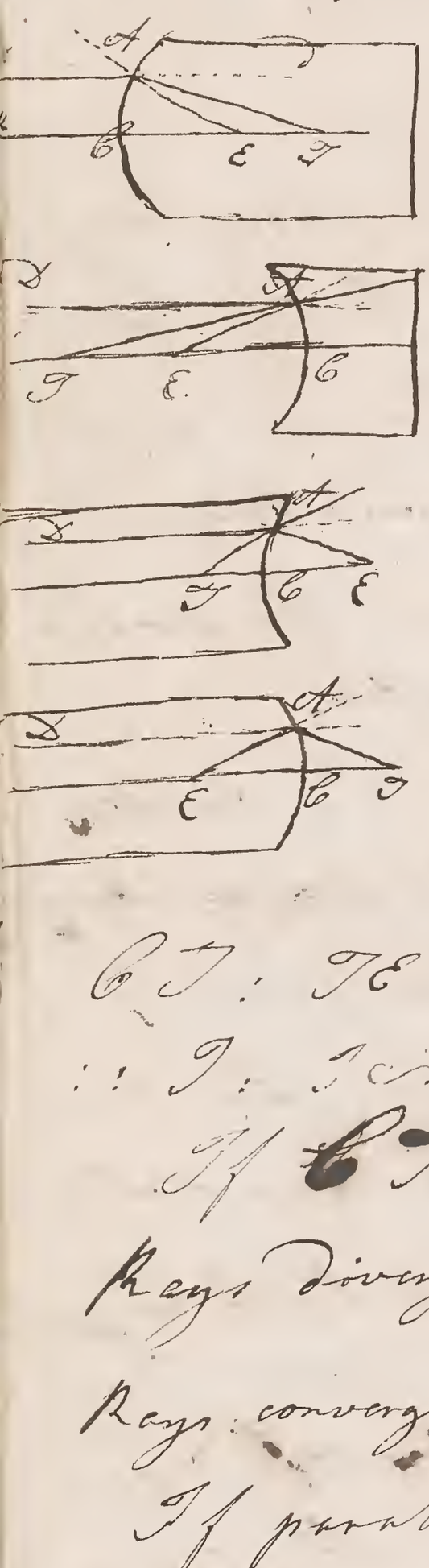
The line 2 q joining the foci of the incident and emergent rays is parallel to g H joining the points which the perpendiculars meet the sides as a g is to g 2 and a H to H g or m to n.

An image formed by the refraction of a medium terminated by two parallel planes will be equal and similar to the object, but will be nearer to it by $\frac{m-n}{m}$. because 2 q : D E :: m - n : m and $2q = \frac{DE \times m-n}{m}$, but D E is given.

The images formed by the refraction of a prism K may be found by taking the points g, if it is evident that the image will be neither equal nor parallel to the object.

Refraction at spherical surfaces.

If parallel rays are incident upon a spherical refracting surface, the foci of refraction will be found in the axis of the sphere and its distance from the surface will be to the radius as the sine of incidence to the difference of the sines of incidence and refraction.



Let A F be a spherical refracting surface, E its center K E its axis and D A one of the rays parallel to K E incident at the point A join E A; let I be the intersection of K E & the refracted ray A I; B I : B E :: I : I sin R

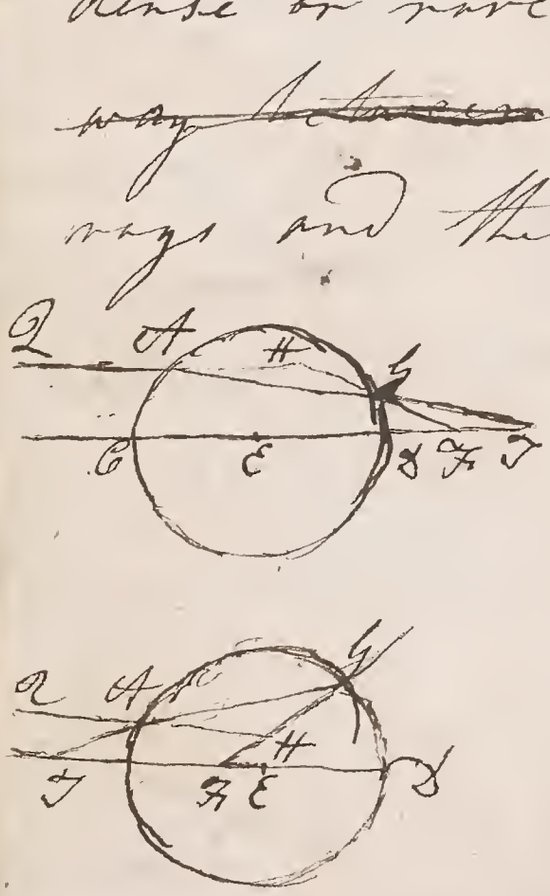
= The sine of the angle A E D is equal to the sine of incidence and of E A I to the sine of refraction = sin R, but A I or (when D A is very near K E) B I is to I E as sine ∠ A E D to sine ∠ I A E therefore B I : I E :: I : I sin R and B D : (B I sin I E =) B E : : I : I sin R

If B I : B E :: I : I sin R it is evident that

Rays diverging from I upon the { Convex / Concave } surface of a { Denser / Rare } medium are refracted parallel

Rays converging to I upon the { Concave / Convex } surface of a { Denser / Rare } medium are refracted parallel

If parallel rays are incident upon a refracting sphere, either dense or rare, the foci of the emergent rays will ~~be found half~~ way between bisect the distance between the foci of refracted rays and the opposite surface.



Let A B G be a sphere, E its center, B D a diameter produced, let 2 A be one of the parallel rays refracted in A G and emergent in G F; the point F, the intersection of G F & B D bisects D I, let 2 A & G F meet in H, then as the angles of Deviation at both surfaces are equal the sides of the triangle A H G are equal and G F I is similar & therefore G F (which is ultimately I D) is equal to F I consequently D I is bisected in F.

A Lens is a transparent body of a different density from the ambient medium and terminated by two surfaces either both spherical, or one plain and the other spherical.

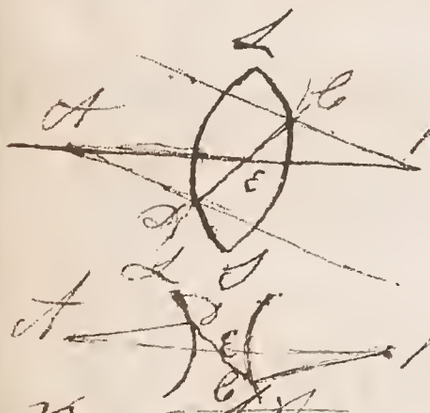
A plano-convex lens has one side plane and the other convex

A plano concave has one side plane and the other concave

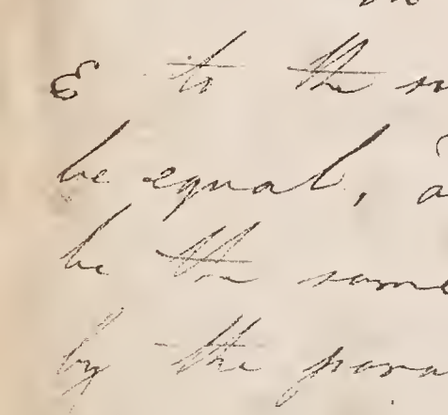
A double convex lens has two convex surfaces

A double concave has two concave surfaces.

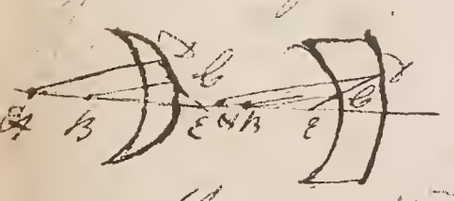
The center of a lens is that point through which every ray that passes is incident and emergent parallel; the center of a lens is found by drawing two right lines parallel through the centers of the two spheres and joining their intersections with the surfaces of the lens; the point in which this line meets the axis is the center.



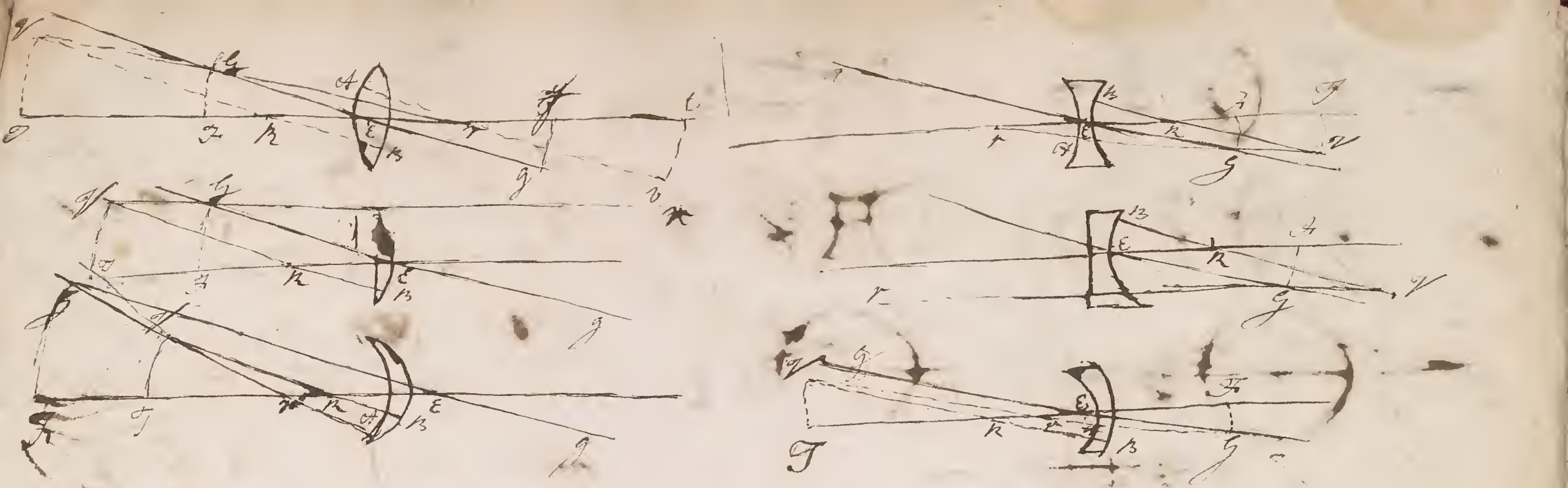
Let L S be a lens and A and B the centers of the surfaces L E S & L D S; let A D & B E be two parallel lines drawn through the centers meeting the surfaces in D & E, let D E be the line joining these points & let C be their intersection parallel the angles A D L & D E B are equal, but these are the angles of incidence because A D and B E are radii of the spheres; therefore the ray D E will be emergent parallel.



The centers of plano-convex and plano-concave lenses are found in the vertices of their axes; for if any line be drawn through C to the surface L E S it is evident that the angles at both sides will be equal, and if the line be drawn to the plane side L S it will be the same as if the ray passed between two parallel planes L S and the tangent at C.



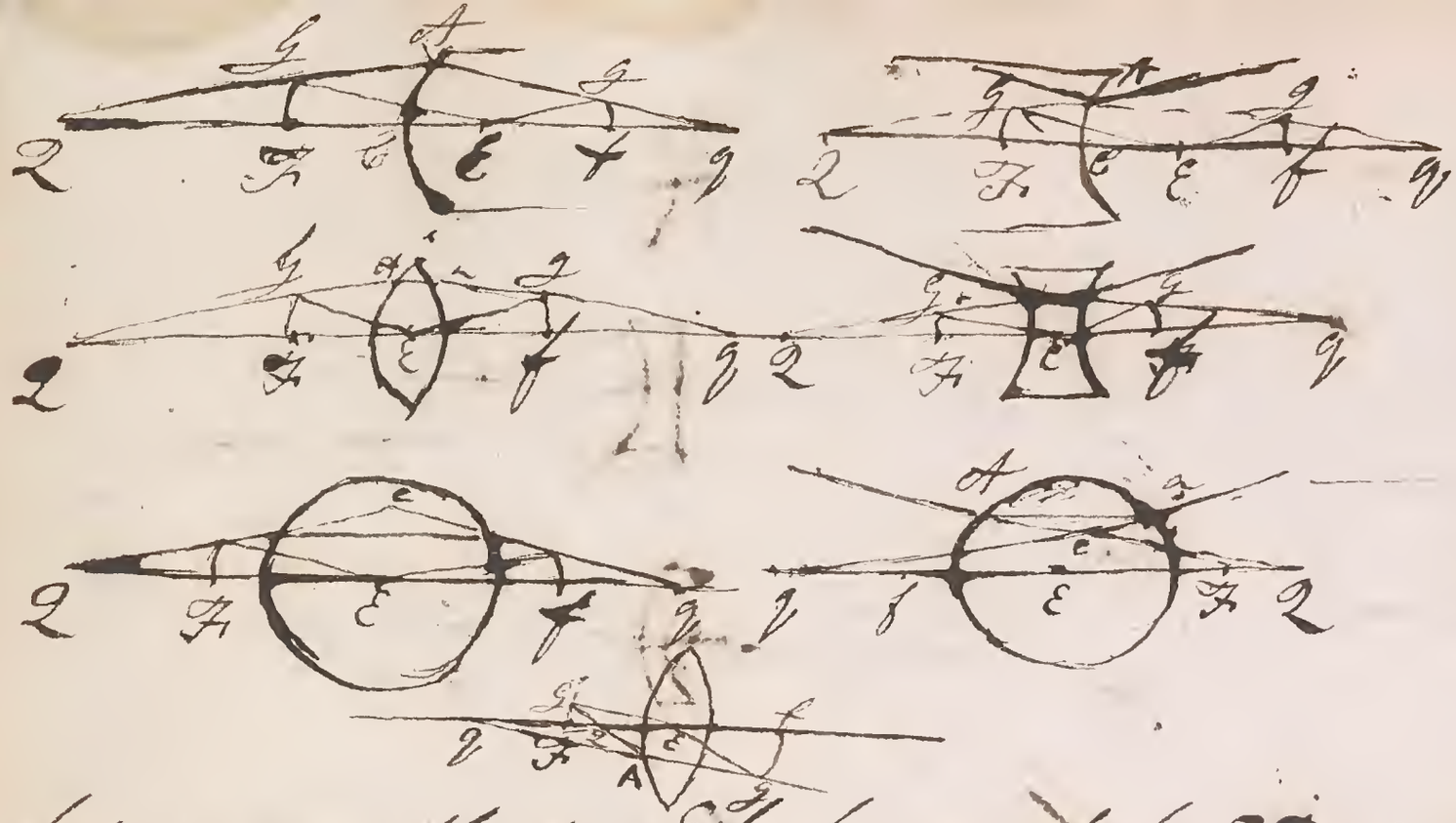
In a concavo-convex lens and in a meniscus the center is without the lens, and is found by drawing two parallel diameters on the same side of the axis; and producing the line joining their intersections with the curvature it will meet the axis; C will be nearest that side which has the greatest



To find the focus of parallel rays falling nearly perpendicularly upon any given lens.

Let AB be a lens, C its center R its axis and R & r the centers of the surfaces; through C draw fCg parallel to the incident rays and through R draw RB parallel to the same; then, as the rays are incident upon the surface B, their focus after refraction at ye first surface will be in RB and RB : Rr :: R : Rf. join fr and let him will meet fC somewhere in f; (because fR is parallel to fC and R & r can never coincide) f is the focus required. For the focus is evidently in the line fr joining f the focus of rays incident upon the surface A and r the center of that surface; it is likewise in fCg as that ray which passes through C may be considered as emergent in the same line, the thickness of the lens being inconsiderable; therefore f the intersection of those lines is the focus. Because fR & fC are parallel, the triangles fRr & fCr are similar; therefore fC : fR :: Cr : Rr and, when the incident rays are parallel to the axis, fC the focal distance has the same ratio to fR; for VR is always a given quantity as it has a given ratio to RB & fC has a given ratio to VB therefore in a given lens fC & fR or FC & CR are given.

The distance from the focus to the center is the same on whichever side of the lens the rays fall; for CF is equal to $\frac{CR \times Cr}{Rr}$ and tC to $\frac{tr \times CR}{Rr}$ but the rectangles CR × Cr and $\frac{Rr}{tr} \times CR$ are equal as $\frac{CR}{Rr}$ is to CR & $\frac{tr}{Rr}$ to Cr in a given ratio. In a double-concave or double convex lens made of glass the sum of the radii (or in the menisci their difference) is to one of them as double the other to the focal distance; because fC is to CR & tC to Cr as 3 to 2.



Let E be the center of any spherical surface sphere or lens, ab, 2 C E its axis & F its principal foci; 2. a rays coming in a direction contrary to those whose focus is 2; let f be between F & C, so that E f be equal to E F, in the sphere and lens and to F C in the surface, then f will be the focus of parallel rays after refraction coming in the same direction. (for in the surface E f : f b :: F b : F E.). If rays diverging from 2 or converging to it are incident upon the surface a b or pass through the sphere or lens a b; their foci after refraction will be g, and 2 F will be to F E as E f. to f g. From the center E describe two arcs F g & f g, meeting one of the rays 2 a at g in f & g, join f E & g E; then as g & F are equidistant from E, and as rays diverging from F are refracted parallel to F E, rays diverging from g and incident upon a b will be refracted parallel to g E, let g a be one of those rays, it will be refracted in a g parallel to g E, for the same reason g a diverging from g will be refracted in a g parallel to g E, therefore the triangles 2 b E & E g g are similar and 2 g : g E : E g : g g or ultimately 2 F : F E :: E f : f g.

when 2 is at an infinite distance; g is at f

when 2 is at a finite distance to the left of F, g is at a finite distance to the right of f

when 2 is at F, g is at an infinite distance

when 2 is between F & C; g is to the left of F

when 2 is at C; g and 2 coincide

when 2 is between C & E; g is between C & 2

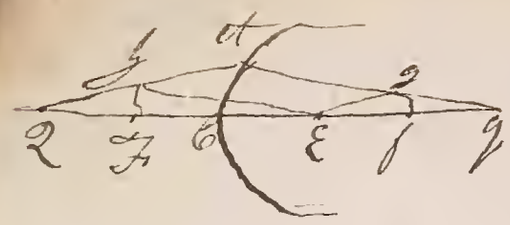
when 2 is at E; g is at E

when 2 is between f & E; g is to the right of f

when 2 is at f, g is at an infinite distance

when 2 is to the right of f; g is to the left of F

when 2 is at an infinite distance to the right of f; g is at F.

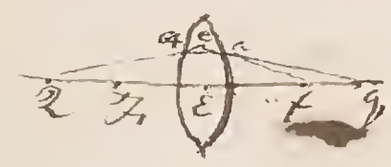


The focus of a pencil of rays diverging from
or converging to a point upon a refracting surface
may be found by this rule $2F : Fb :: bf : fg$.
because $2F : Ef :: FE : fg$ and Ef is equal Fb & EF to bf.

The focus g may be found by this rule. $2F : 2C :: 2E : 2g$.
for the triangle $2gE$ & $2Cg$ being similar ($2g : 2C$ that is)
$2F : 2b :: 2E : 2g$.

The rule for finding the focus in a sphere or lens is

$2F : 2E :: 2E : 2g$ for, producing the incident
and emergent rays till they meet in e the triangs



$2gE$ & $2eg$ will be similar and $2g : 2C$
that is $2F : 2E :: 2E : 2g$.

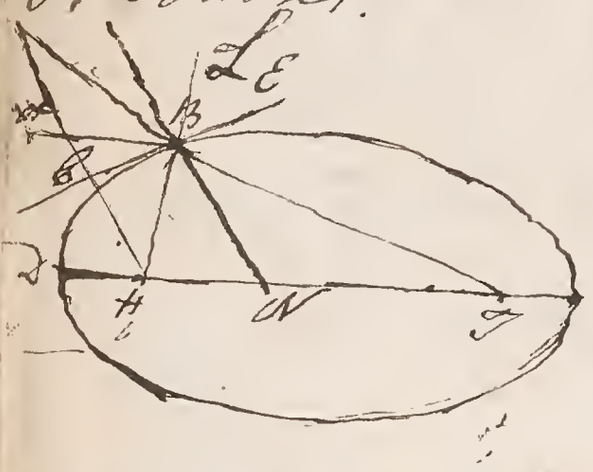
The distances $2F$ & gf are inversely as each other
and 2 is always on the contrary side of F that g is of f
therefore if one be put in motion the other will move the
same way though with a very different velocity.

When 2 & g are on the same side of the refracting surface
if the incident rays flow from 2 the refracted rays will
flow from g; if the are on contrary sides rays flowing from 2
will be refracted to flowing g; and vice versa; because rays
always forward.

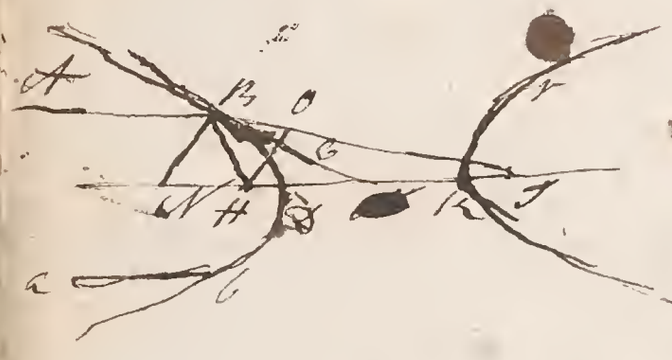
The focus g thus found is not the point in which all the rays
meet, but only the intersection of the two next to it through
the center and that nearest to

All lenses having the same focal distances when put in one
anothers places will have the same effect as the focal Distances
and not the diameters of the spheres, determine the situation
of foci.

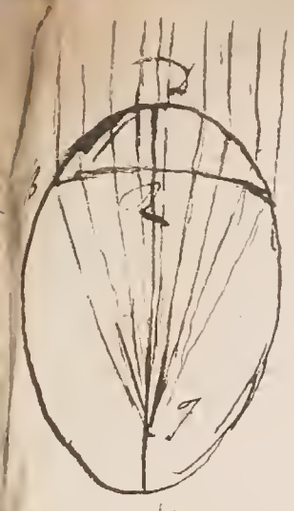
Every ray belonging to a pencil of rays may be made to converge to or diverge from a given point accurately, by the conic sections.



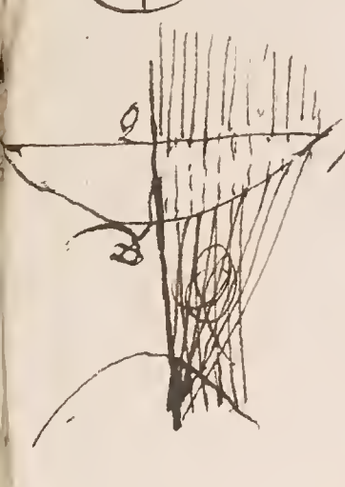
Let D B K be an ellipse and B D G K r a opposite hyperbolas and in the Ellipse let D K the major axis be to H F the distance between the foci as the sine of incidence upon a dense medium to the sine of refraction, and let D K in the hyperbola be to H G as the sine of incidence upon a rare medium to the sine of refraction; let solid be generated by the revolution of these sections about D K; Rays as A B coming in a direction parallel to D K and incident upon the convex surface of the spheroid and upon the concave surface of the hyperboloid will be refracted converging to F.



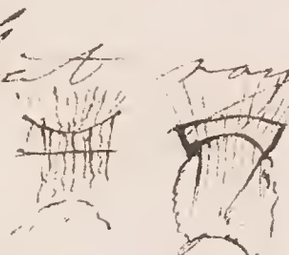
Let A B be one of those rays incident at the point B, join B F & B H, draw L B a tangent at B and now H N at right angles to it meeting the axis in N; through H draw H O parallel to B F meeting F B produced in O. Then as the triangles O B G & H B G are equal and similar (the angle O B H being bisected) B H & B O are equal and F O equal to F B ± B H or to D K; A B N is the complement of the angle of incidence or its complement to two right angles therefore the sine of B N F which is equal to A B N is the sine of incidence; but the sine of the angle B N F is to the sine of N B O as B F to N F that is (as F O to F H or) as D K to F H) the incidence to refraction therefore A B F is the angle of refraction and consequently B F the direction of the refracted ray; in the same manner it may be proved that every other ray will pass through F.



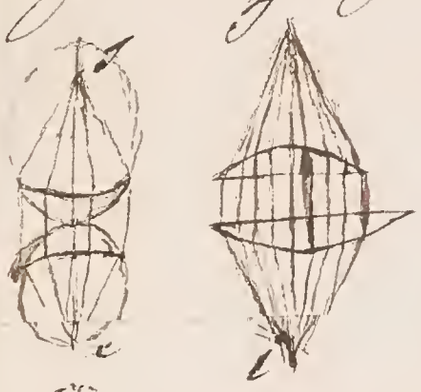
To make parallel rays converge to a point; Let BDB
be an ellipse and BDB an hyperbola; from the center
I describe any arc of a circle with as radius less
than ID in the ellipse and in the hyperbola draw
any line BQB meeting the section in B & B
the lens formed by the revolution of BDQ round
its axis QI will refract all rays parallel to
QI ~~incident upon the ellipt~~ to the focus I.



It is evident that rays diverging from I will be refracted
parallel to DQ.

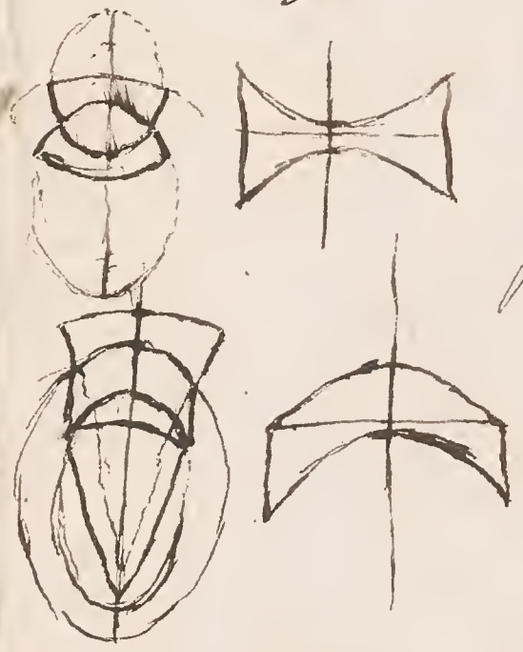


If any line be described as before but with a radius greater
than ID and a line drawn as before but not meeting the
hyperbola will make parallel rays diverge from I, and
rays diverging from I emerge parallel to DQ



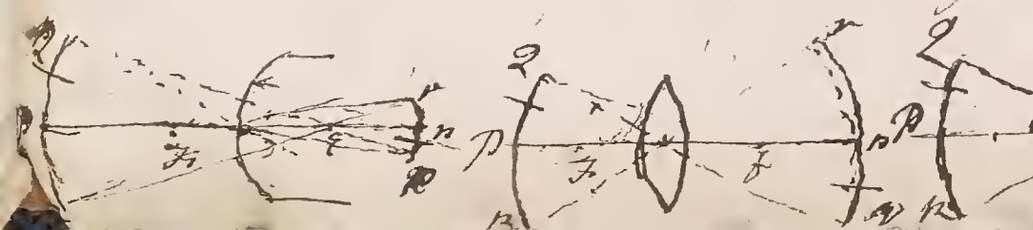
Two convex elliptic or hyperbolic lenses whose axes are parallel
and their convex sid towards one another will make
rays diverging from converge to i.

If the concave surfaces be placed in the same
manner; converging rays will be made to diverge
from a given point.



Rays diverging from one point may be
made to diverge from another by two elliptic
or to hyperbolic glasses, the one concave
and the other convex.

If an arc of a circle described about the center of a refracting
spherical surface sphere or lens be considered as an object
its image will be a similar concentric arc whose length is to that
of the object as their distances from the center. Because the principal
foci of rays coming in any direction are equidistant from the center
and the focus of rays diverging from a point is determined by the
distance of that point from the center and the focal length
therefore the focus after refraction of every point is in a circle concentric
with it and the axis; the image is as the distance, because all similar
arcs are as the radius



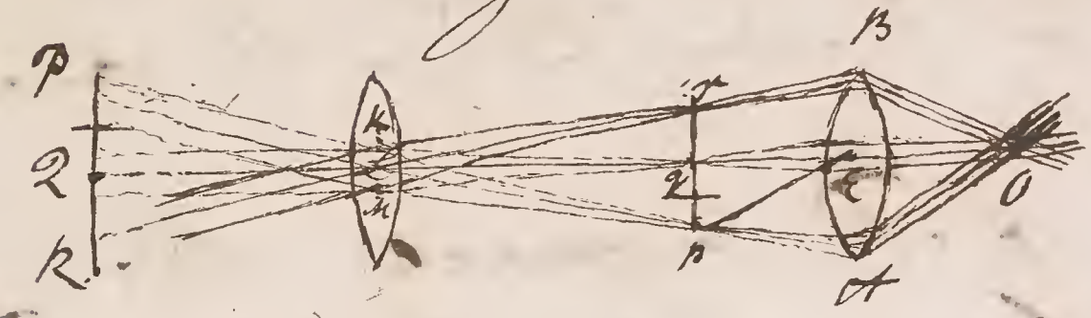
If the image is the same
side with the object
it is erect & vice versa

Thompson.

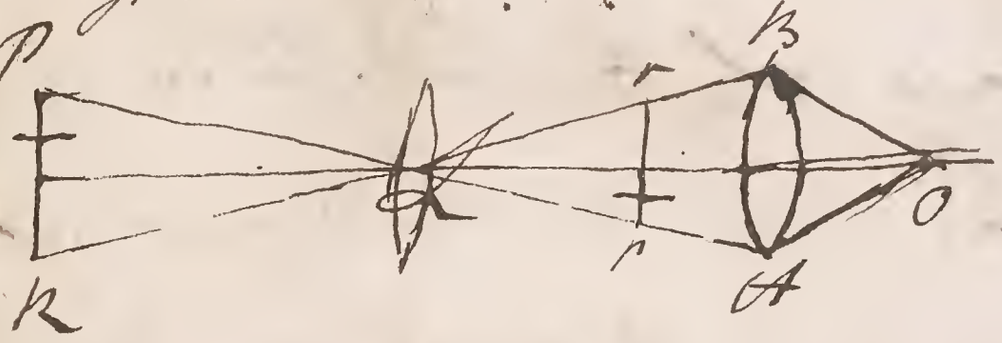
Sir James Hall Bart.

Telescopes.

1st The astronomical telescope with two double convex lenses only. Let K L M be the first lens or the object glass and at L B the second or the eye glass; let them be so placed that their principal focii coincide at q and let L E be the sum of their focal lengths; an eye placed at O (the principal focus of E on the contrary side with q) will see an inverted image of an object P Q R, placed beyond M L K, and the apparent magnitude of this image will be to that of the object, with the naked eye, as L q to q E that is as the focal length of the object-glass to the focal length of eye-glass. For suppose the object to be at a great distance (i) the rays diverging from every point in it may supposed to be incident parallel upon M L K; they will therefore form an image (p q r) passing through q which will be inverted being on a contrary side of L with P Q R; then as the lenses E & L have a common principal focus, the rays which cross one another at the points p, q & r will be refracted parallel by the lens E and the axes of the pencils will meet at a point on the axis a little beyond the principal focus of E (by supposing L to be a focus of incidence), therefore an eye placed at O will see the image q p under the angle E O A or (by drawing p E to the center of the lens) q E p; and the angle under which the eye at O sees the object P Q is P O R or P L Q or its alternate p L q; the angles p L q & p E q having a given subtense p q, are to one another inversely as the radii L q & q E; therefore &c. Q E D.

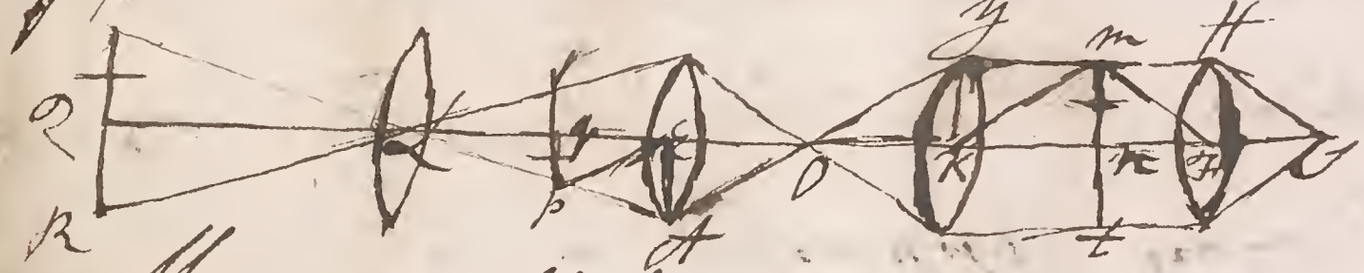


When the focal length of the object glass is greater than that of the eye glass the image will appear greater than the object & when the focal length of the object glass is the least of the two the object will be magnified in proportion.



The field or space which this telescope takes in, is measured by the angle A O B which the eye glass subtends at the center L of the object glass; for, if L P & B L K being the axes of the extreme pencils, every pencil which incident between A & B must diverge from some point between K & P.

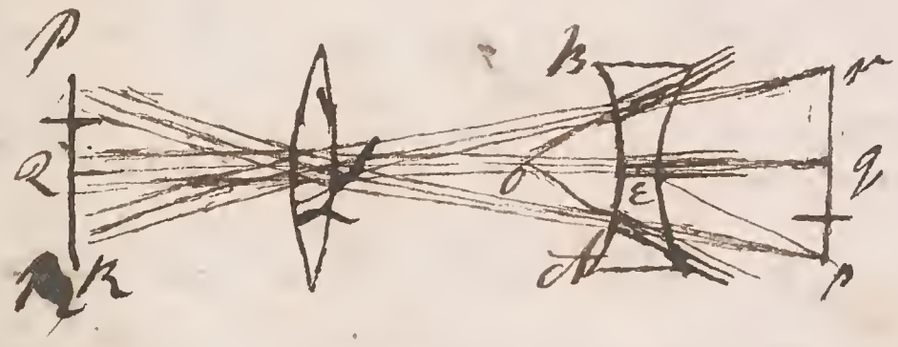
The second kind of telescope is the same with first only that in this the object is seen erect by the addition of two other lenses.



Let the lenses F & E remain as in the first telescope; then, as before, the parallel pencils of rays will cross at O; let a lens whose focal distance is equal to that of E be placed beyond O to intercept them; after passing through this they will form an image m n t at the principal focus of the lens F, which will be in a situation contrary to that of the last image and the same with that of the object; beyond this image let another lens be placed whose principal focus coincides with that of F, and by this the pencils which form the image m n t will be refracted parallel and will cross again at K as they did before at O; if the focal lengths of the lenses K & F be equal to that of E An eye placed at L will see the object erect and its angle it subtends, that is, its apparent magnitude will be equal to that in the first case. If the focal length of F only be diminished the image will be to the object in the ratio compounded of L y to y and of K n to E to n F.

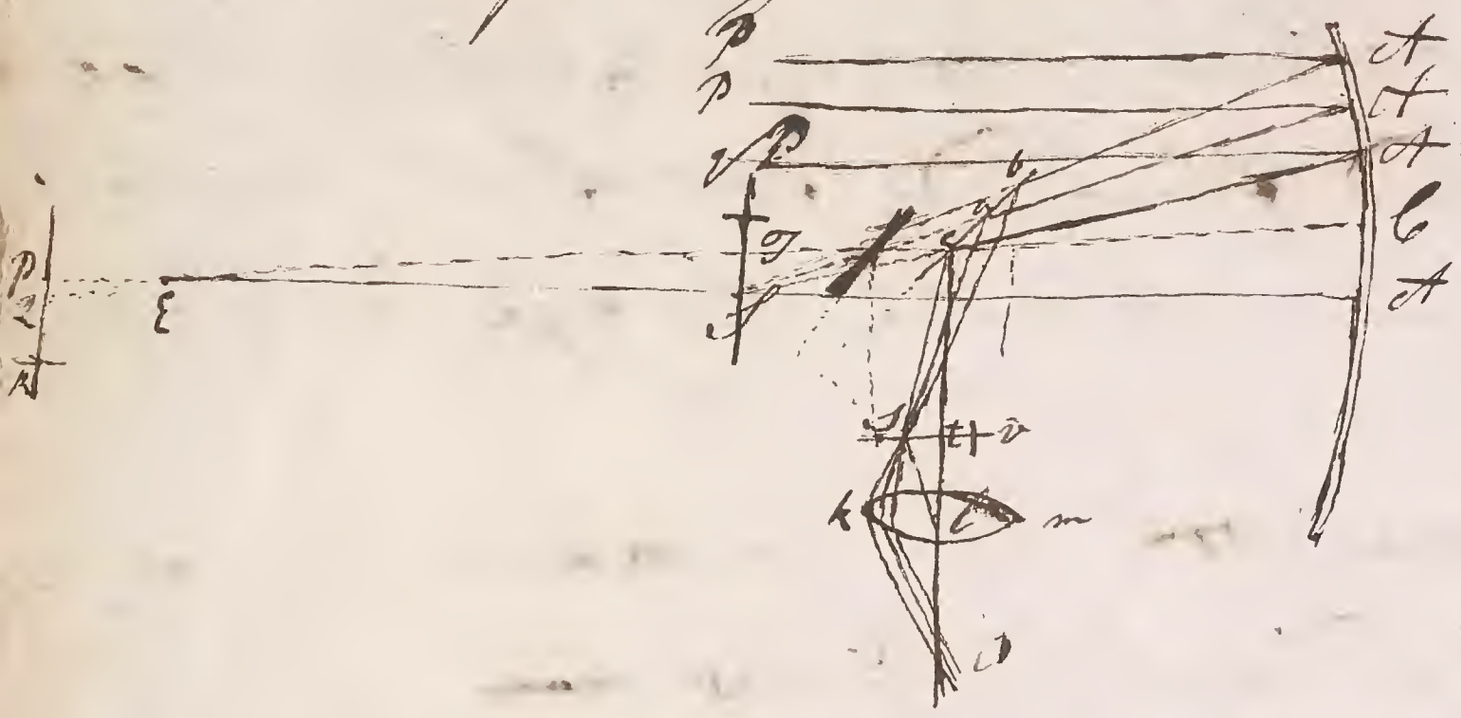
As all eyes are not able to unite parallel rays accurately upon the retina the eye glass is made to move backwards and forwards for if it be pushed in or drawn out rays will enter the eye diverging or converging.

The Galilean telescope or the common perspective glass is composed of a double convex and a double concave lens. The double concave lens is



placed so that its principal focus coincides with that of the convex and the distance between them the difference of the focal lengths; The pencils of rays coming from the object, and which are refracted by the lens L to form an image at pqr are intercepted by the double concave lens AEB which they will be refracted parallel to their respective axes, and if the point L be considered as the focus of incidence of these pencils upon the lens AB they will diverge after refraction from some point O in the axis; therefore if an eye which is able to unite parallel rays upon its retina be placed very near the center of the lens E, it will see the object PQR erect and the apparent magnitude of this image will be to the apparent magnitude of the object to the naked eye as Lq to Eq; For the angle under which pq is seen is EOA or its equal qEp which is to qLp the measure of the apparent magnitude of the object in the inverse ratio of the radii Eq & Lq (the substance qp being given) that is as Lq to Eq or as the focal length of the object glass to the focal length of the eye glass. The field of this telescope is measured by the angle which the pupil of the eye subtends at the center L of the object glass; for, as the pencils diverge after the last refraction the field must depend on the magnitude of the pupil.

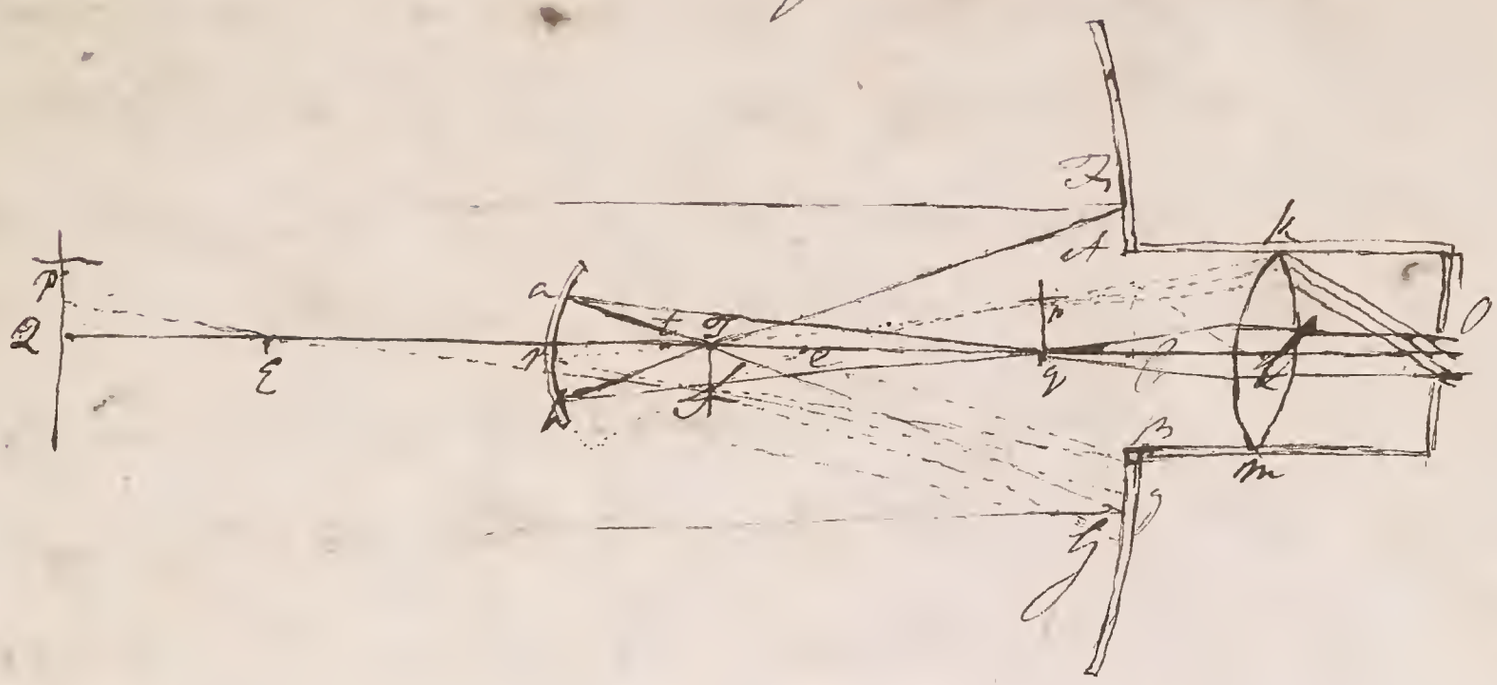
The reflecting telescope is invented by Sir Isaac Newton.



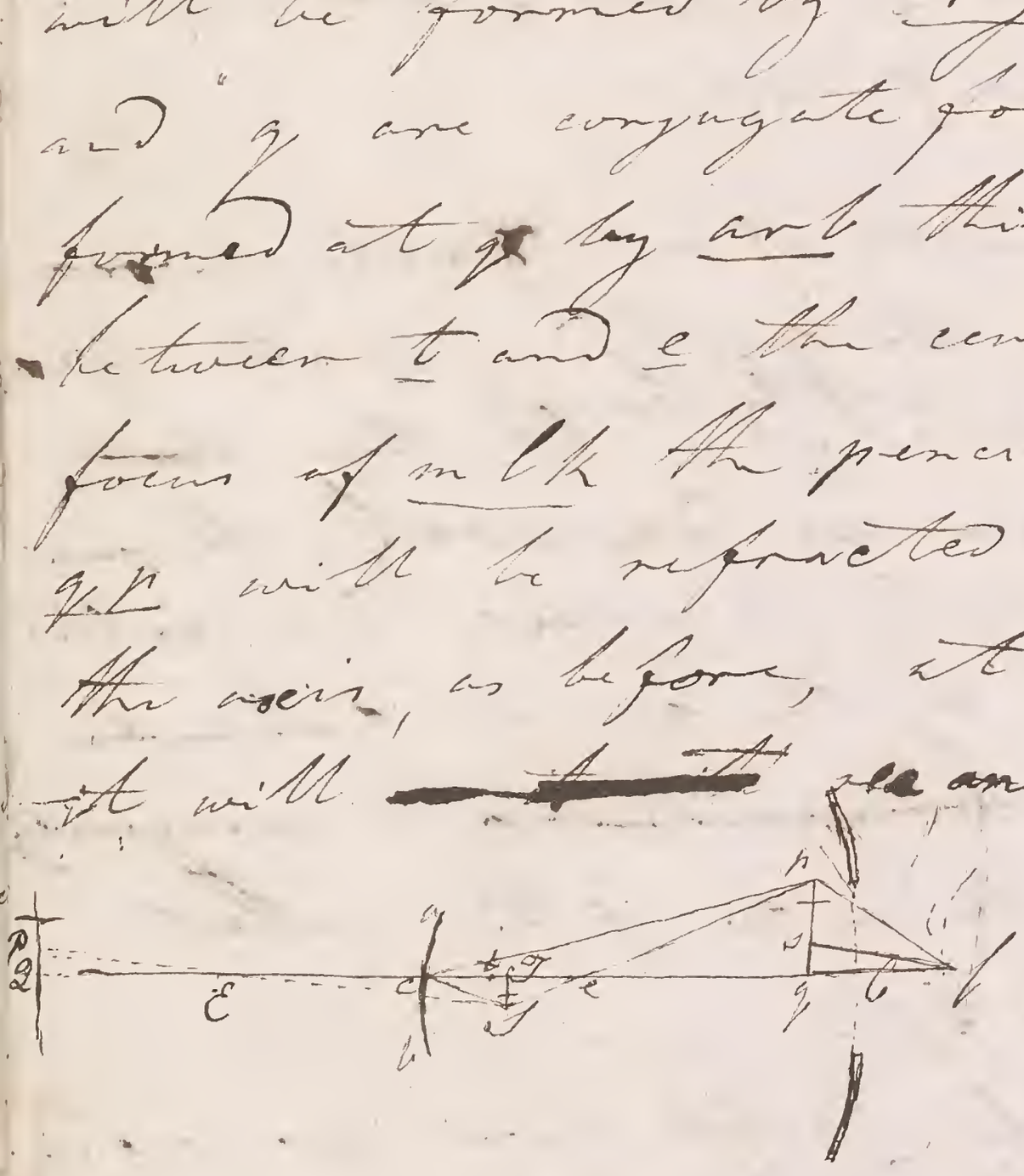
Let I J be the image of a remote object P 2 formed by reflection from a spherical speculum A b A; let P E A and 2 E 2 b be the axes of the extreme pencils; then as the object and image are on different sides of the center E the image will be inverted. Let a glass speculum a b c be placed so as to intercept the rays converging to form the image at I; then an image will be formed at 2, at the same distance beyond that other was behind the glass; Let a double convex lens be placed at 2 so that its principal focus shall coincide with 2 or I; then, as in the other telescope, the pencils will be refracted parallel and will cross one another at a point in the axis, O, near the principal focus.

The apparent magnitude of the image is to the apparent magnitude of the object, with the naked eye as the focal length, or half the radius, of the speculum to the focal length of the lens; Draw O k from O to the center of the lens, this is evidently parallel to the emergent pencil k e; then the image o t is seen by an eye at O under the angle k O t or e b t and o b t is to e E 2 as E 2 or E I is to b t (because the subtenses e t & I 2 are equal) that is as the focal length of the speculum to the focal length of the eye glass.

The Gregorian Telescope



Let A B be a concave spherical speculum having a circular hole at B; let E F g b be an axis passing through the center of the hole, & let F be the principal focus in that axis; & let another concave speculum a r b be placed so that rays diverging from F may be reflected by it converging to to g, the principal focus of a double convex lens m k k placed upon the same axis behind the hole & let t be the principal focus of a r b. Let p 2 be a remote object; then as ray diverging from it may be supposed to be incident parallel on image of it will be formed by A B at F inverted; then as F and g are conjugate foci a second image will be formed at g by a r b this will be erect as F is evidently between t and e the center of a r b; as g is the principal focus of m k the pencils which formed the image g L will be refracted parallel and will enter in the eye, as before, at D where if the eye be placed it will [illegible] an image of the object p 2 erect.



The apparent magnitude of the last image is to the apparent magnitude of the object to the naked eye as the (a D to p E) x p E to p E 2 or D E F; take g L equal to F D and join r L; then as the angles D E F & r L g have equal subtences they will be inversely as the radii and the angles r L g & p L g having a common radius will be as the subtenses; that is D E : F L :: g L : (E D or) F L, and r L : p L (:: (g a) D F : p g :: D e : e g) :: t e : t g; therefore p L g will be to D E F, that is, the image will be to the object in the ratio compounded of t g to t e and of F L to g L; that is if t g be taken equal to F L as F L 2 to t e x g L.

The apparent magnitude of the last image is to the apparent magnitude of the object to the naked eye as the (a D to p E) x p L g to

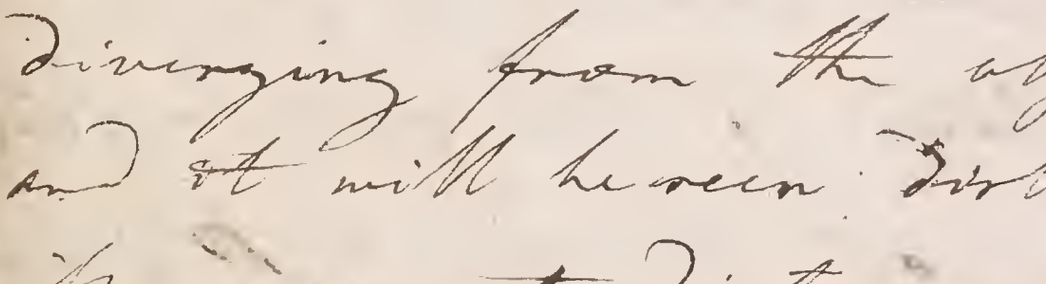
A Telescope may be made in the same manner with the last and having the small speculum convex and placed between F and C to receive the rays converging to F; this is shorter than the other by twice the focal length of the small reflecter; and the object will be seen inverted.

The field of vision in these telescopes is measured by the angle which the eye glass subtends at the vertex of the small speculum; for if the distance be given, the images and consequently the real object will vary with the angle which the lens subtends.

To accomodate this telescope to a near distance the small speculum must be moved a little further from the great one; for when a remote object approaches its image at O approaches to F and when Of is diminished its reciprocal $\frac{1}{Of}$ is increased.

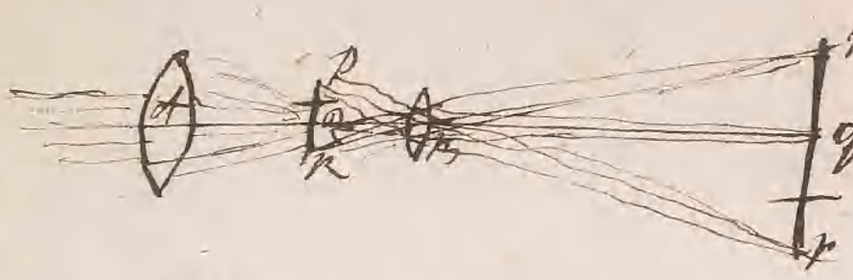
To make the pencils emerge diverging or converging the small speculum must be moved a little nearer to or further from the great one

A simple microscope consists of one double convex lens as L placed between the eye at O and a small object P.2R, passing through the principal focus of the lens and which, 2O is less than the least at which the eye can see distinctly; the pencils diverging from the object will be refracted parallel and it will be seen distinctly by the eye at O, thus its apparent distance with the increased and consequently its apparent magnitude increased in the ratio of the smallest distance of distinct vision to 2L.



The double microscope consists of two double convex lenses; let F be the focus of the lens A and let a small object PQR be placed a little beyond F; an image of this will be formed at q; let y be the principal focus of the lens B then the pencils which formed the image pqr will be refracted parallel by B and will cross the axis at L; then $\frac{Of}{fq} : \frac{qf}{fB} :: \frac{Of}{fB} : \frac{fO}{fB}$ that is $\frac{L\,O\,k}{L\,A\,k} : \frac{L\,A\,k}{L\,A\,k}$ the apparent magnitudes of the image an object;

The solar ~~microscope~~ microscope; Let A be a double



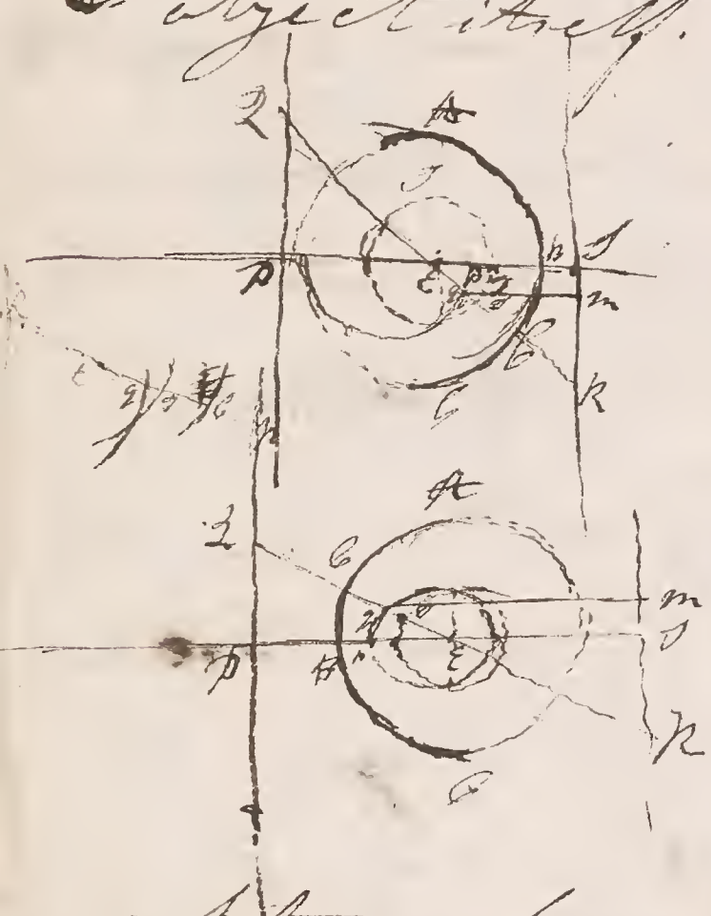
convex lens upon which the rays of the sun fall parallel to its axis; let a small transparent image PQR be placed near the focus of this lens and let B be a double convex lens placed so that its principal focus shall fall between Q & B then a magnified image of PQR will be formed at q which may be received upon a white wall.

The magic lantern is the same as this only that a candle is used instead of the sun and its rays are collected upon the ~~may~~ object by a convex lens before or a concave speculum behind.

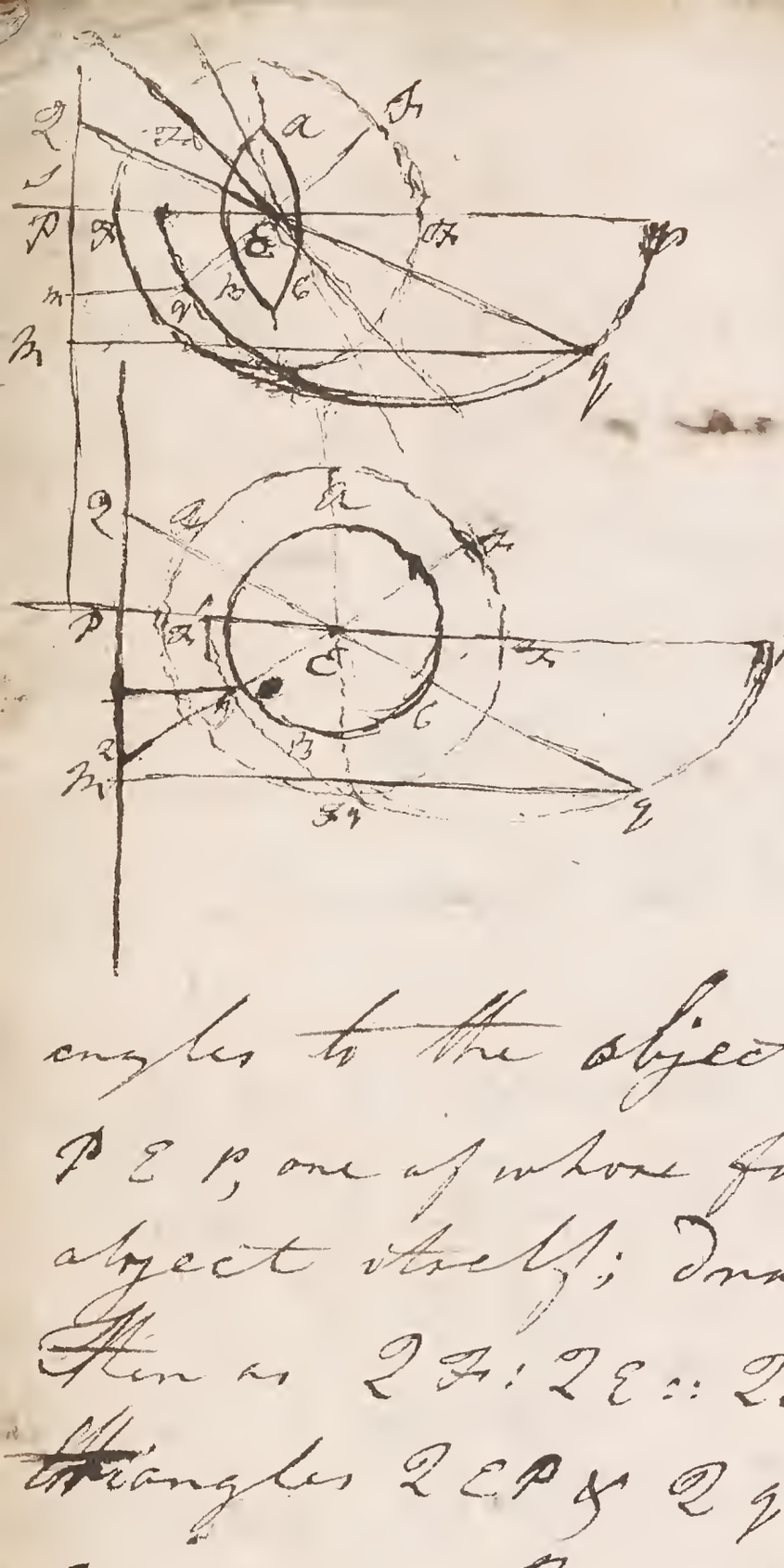
The camera obscura is a single convex lens which ~~coll~~ brings the pencils diverg from objects to an image which may be received upon a sheet of white paper the image is inverted and may be made erect by the addition of two other lenses; by placing a plane speculum to intercept the rays, before they have formed the image, it may be received upon an horizontal plane.

the The image

def 1 of an infinite straight line, formed by reflection from a complete circular specutum or by refraction through a spherical surface, sphere or lens, is one of the conic sections whose axis is at right angles to the right line, one of whose foci is the center of the sphere or lens & the directrice to that focus a right line at an equal distance beyond the center parallel to the object, in reflection, but in refraction the object itself.



Let ABC be a circular specutum, E its center, E I all the radius I P Q an infinite right line; let P E I be a diameter at right angles to P Q meeting it in P and taking E I equal to E P draw O R parallel to P Q. y will be a point in a conic section whose focus is E and whose directrix is R; through y draw y m at right angles to I R; produce Q E y (the axis of the rays diverging from Q upon C) to R; then as E R is equal to Q C and Q I : I E : : I E : I y by division Q E : E y : : Q O : I E and (Q E - E y) R y : E y : : (Q I - I E) Q E : I E; because the triangles E I R & y m R are similar | y m : R y : : E I : (E R) Q E therefore by compos. y m : E y : : E O : I E that is E y is to y m in a given ratio; but it is the universal property of all the conic sections that the line drawn from any point of the surface or surfaces to one of the foci has a given ratio another line drawn from that same point at right angles to the directrice of the focus; therefore y is a point in the conic section &c. The same may be proved of any other point y therefore the locus of y is a conic section which will be complete as lines may be drawn through E from P Q, which is infinite, in every possible direction and as the ratio is estimated both from the convex and concave surfaces, every point will have a corresponding point y. Q.E.D.



Let ABC be a refracting sphere or lens whose center is E and its focal length EF and 2Pm an infinite straight line every point of which rays are supposed to converge as well as diverge upon the sphere or lens (as a straight line may be considered as a circle of an infinite radius) Let PEp be a line passing through the center and at right angles to the object; q will be in a conic section whose axis is PEp, one of whose foci is E and the directrix to that focus the object itself; draw qm at right angles to the object, then as 2F : 2E :: 2E : 2q, & 2E : 2q :: FE : Eq and because the triangles 2EP & 2qm are similar 2E : 2q :: Ep : qm, therefore Eq : qm :: FE : EP that is Eq has a given ratio to qm; it may be proved as in the first part that q is a point in a section whose focus is E and directrix 2Dm. and that the locus of q is a complete conic section. 2-E D.

Cor. 1st The parameter of the section is equal universally to twice the focal length; because if through E a line be drawn perpendicular to the axis, the points in which it meets a circle described about E with the radius EF or ED will be common to the section and that circle* whose diameter, ~~Eq~~ twice EF or ED will then become the focal ordinate and equal to the parameter.

Cor 2d The image of an infinite straight line, whose distance from the center is infinite, is a perfect circle whose center is E and radius EF or EI, for every point being at an infinite distance, Iq will vanish, in the speculum, and the difference between 2F & 2E will become equal to that between 2E & 2q; in the

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*Because, in reflection, when 2E is infinitely great 2D vanishes and in refraction, when 2F & 2E are both infinitely great the difference between 2F & 2E is equal to the difference between 2E & 2

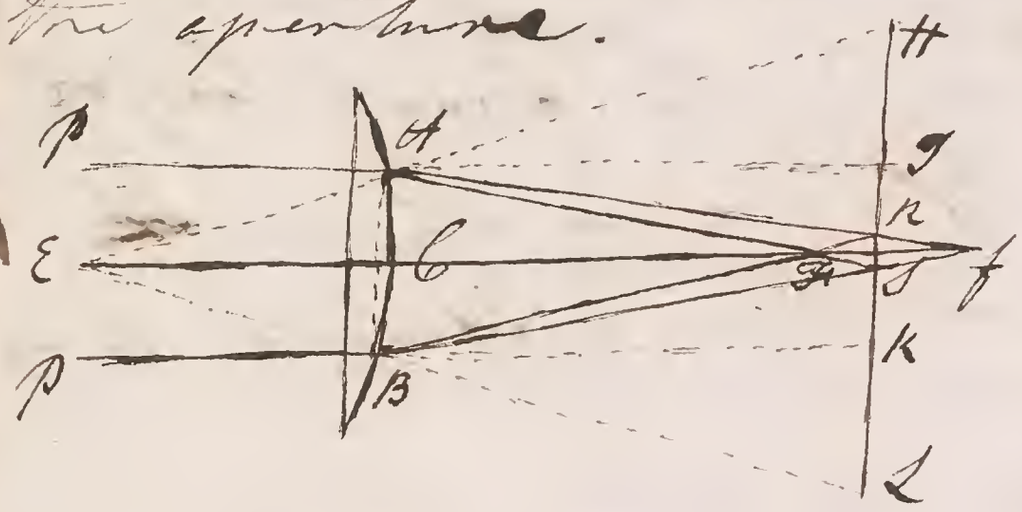
the sphere and lens; when the line is brought to a finite distance from, the image is an ellipse whose vertex nearest to the object is entirely within the speculum when the object is without and without when the object is within; when the object comes to F or F, the image is a parabola; when the object is within F the image is two opposite hyperbolas and the object comes to the center the object and image coincide.

Having the center of the speculum or lens given the image may be found, for let E be the center, P the nearest point of the directrix in the speculum the nearest point of the object, PE a diameter drawn through E meeting the circle described with the radius ED in D, join PD and produce pD to L; make the angle LDe equal to pDE; the intersection D' with the radius will be the other focus of the section; as is evident. By this it may easily be determined what the section is.

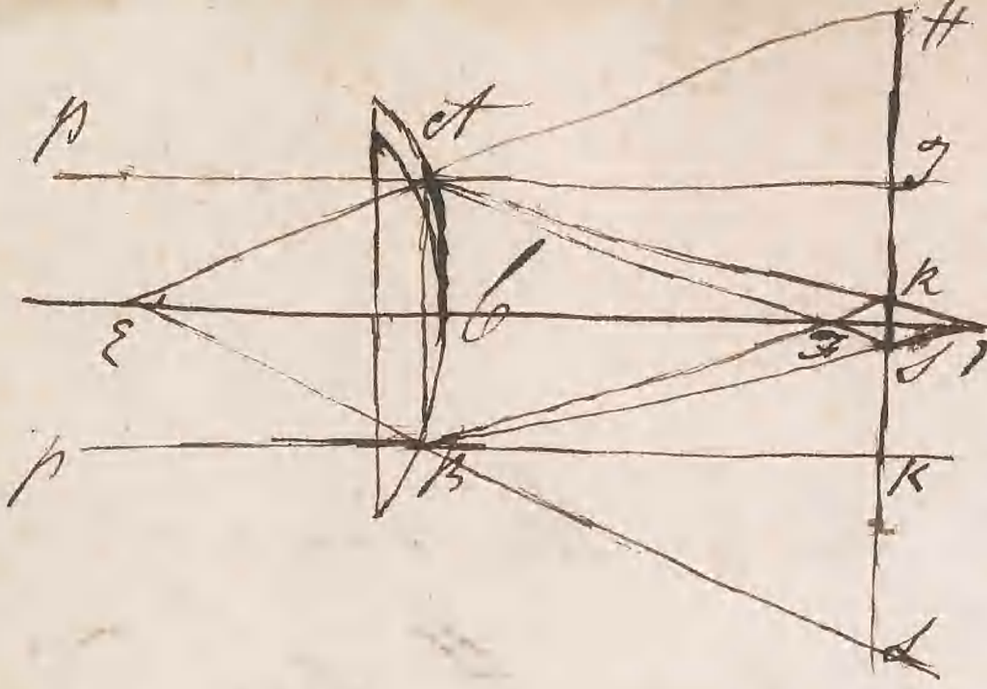
Aberrations.

It is found by experiment that the rays of light are not equally refrangible, that is they form different angles with the perpendicular to a refracting surface, if a pencil of the sun's rays impinge upon a refracting surface it will be split into colours which are supposed to be seven in number, violet, indigo, blue, green, yellow, orange, red, of which the most refrangible are violet and the least red. If the sine of the angle of incidence be as 50 the sines of refraction of the red and violet rays will be as 77 & 78.

If heterogeneal parallel rays are incident upon the spherical surface of a rare medium; the ~~least diameter~~ of the least circle into which every ray can be collected, after refraction, has a constant ratio to the diameter of the aperture.



Let ACB be the aperture of the concave spherical surface of a rare medium, whose center is E, or of a plano convex lens whose plane side is at right angles to the parallel incident rays; Let ECf be the axis parallel to the incident rays ~~and passing through the center of ACB~~ in which let f be the principal focus; Let P & p be a small pencil incident at A nearly perpendicular to the surface and let its violet rays cut the axis in R and its red rays in f; let PB be another small pencil incident at B at the same distance from the axis with A then it is evident that the corresponding extreme rays in each will meet at R & f, let R & S be the points of intersection of the most refrangible in the one and the least in the other; join RS, then its evident that RS is the diameter of a circle through which every ray refracted at ACB must pass.

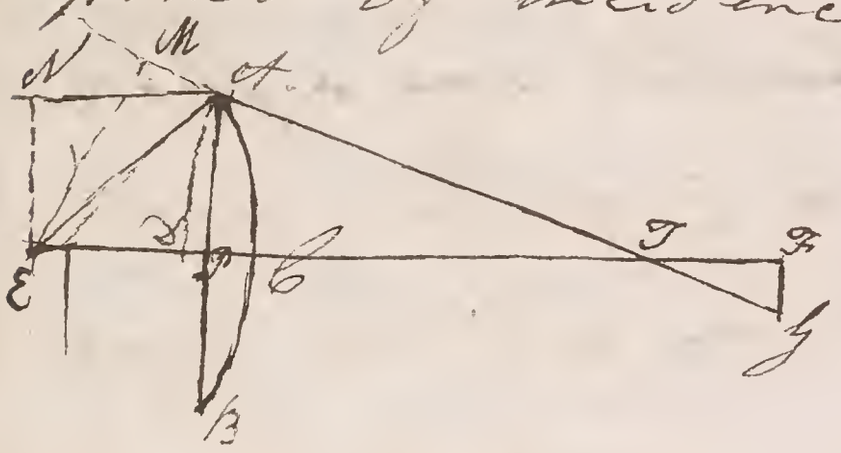


Let I be the sine of incidence K of the least refrangible ray and C of the most refrangible, produce RA till it meets EA & CB produced in H & Land PA & PB in I & K; & as PH & PB fall nearly perpendicular upon the surface, the angles of incidence and refraction will be very small therefore HI, HK & HL may be considered as I, K & L and RL : RI :: L-R : R-I that is, RC (or): IK - RL :: L-R : 2R - 2I and RL : (IK or) AB : : L-R : R+L-2I but I, L & R are constant quantities therefore RL is to AB in a given ratio. Q. E. D.

The diameter RL of the circle through C which all the rays pass is a 55th part of the diameter of the aperture AB for suppose I as 50 then R & L will be as 77 & 77 and the ratio of 2-x to 1+x-2I will be that of 1 to 55.

In different surfaces the angle RAL is as the radius of the aperture directly and the focal length inversely for the angles at A & C are nearly as AC directly and AC & BC inversely and their difference BA will be in the same ratio

If an homogeneal ray is incident, parallel to the axis of a spherical surface or plano-convex lens, it longitudinal aberration caused by the curvature of the surface has a given ratio to the versed sine of the angle between the point of incidence and the axis.



Let A C B be a spherical surface or plano convex lens whose center is E and principal focus F; let A be the point of incidence of a ray A P parallel to the axis and let it be refracted in A Q meeting the axis in Q; draw E M at right angles to A F and E M to F A produced then it is evident that E M & E N are the sines of refraction and incidence; for them suppose m & n then F D will be to P C as $\frac{m^2}{m-n \times n}$..

From the center F with the radius F D describe the circle A S meeting the axis in S; then as the triangles A Q P & E Q M are similar ~~E. Q. : Q A (A P or) E M. E Q~~ ~~:: m. m~~ E Q : (Q A or) Q S :: E M : (A P or) E N :: m : n :: Q E : Q C and by division Q E : E C :: (Q E - E Q or) Q S : (Q C - Q S or) Q S - Q C and $\frac{Q S}{Q C} : \frac{Q C}{Q E} :: Q E : (Q E - E C$ or) $E C :: \frac{m}{m-n} : \frac{m}{m-n}$

As the versed sines of the arcs A C & A D are nearly in the inverse ratio of their radii C P : P S :: Q A : E A that is :: n : m - n. (because A E Q is equal to the angle of incidence and A Q E to the difference between the angle of incidence and refraction) and (Q P + P C or) Q C : P C :: m : n; then, by compounding this ratio with that of Q F : Q C, Q F : P C :: m^2 : $\frac{}{m-n \times n}$.

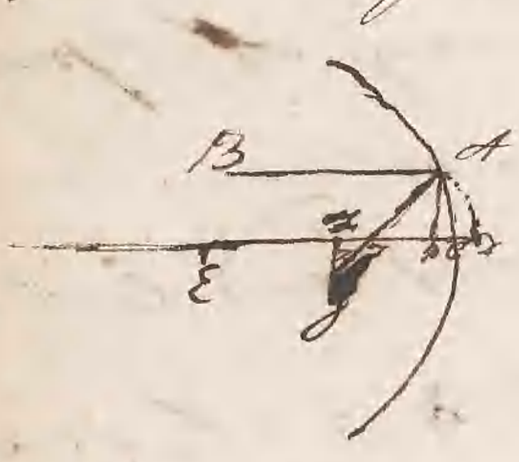
If the segment A C B be considered as a lens the longitudinal aberration of its extreme ray A F will be equal to $\frac{1}{2}$ of its thickness as is evident by substituting 2 & 2 for m & n.

The longitudinal aberration $Q F = \frac{m^2}{m-n \times n} \times \frac{A P^2}{2 C F} =$ $\frac{m^2}{m-n^2} \times \frac{A P^2}{2 C F}$ because P C is nearly equal to $\frac{A P^2}{2 C E}$ and E C to C F $\times \frac{m-n}{n}$.

Draw Z G perpendicular to C F till it meets A Q produced the lateral aberration $Z G = \frac{m^2}{n^2} \times \frac{A P^3}{2 C E^2} = \frac{m^2}{m-n} \times \frac{A P^3}{2 C F}$ for F G : G F :: A P : Q P or C F or $\frac{m}{m-n} \times C E$.

When the semidiameter of the concavity or the focal length is given, the longitudinal aberrations are as the squares and the lateral aberrations as the cubes of the linear apertures.

The longitudinal aberration of a parallel ray reflected from a spherical speculum is equal to $\frac{1}{2}$ of the versed sine of the arch between the incident ray and the axis if the angle of incidence be small



Let A E be a reflecting speculum whose center is E and principal focus F; the aberration D F of the ray B A is equal to half P E the versed sine of the arch A E; From the center F with the radius F A describe a circle meeting the axis in D then as P E & P D to one another as D F & E E, P D will be nearly double of P E and P E, E D equal then D F is equal to (2 E F - 2 E D or E D - E E or E D or) P E.

The longitudinal aberration $D F = \frac{A P^2}{4 E E}$; for $P E = \frac{A P^2}{2 E E}$ nearly

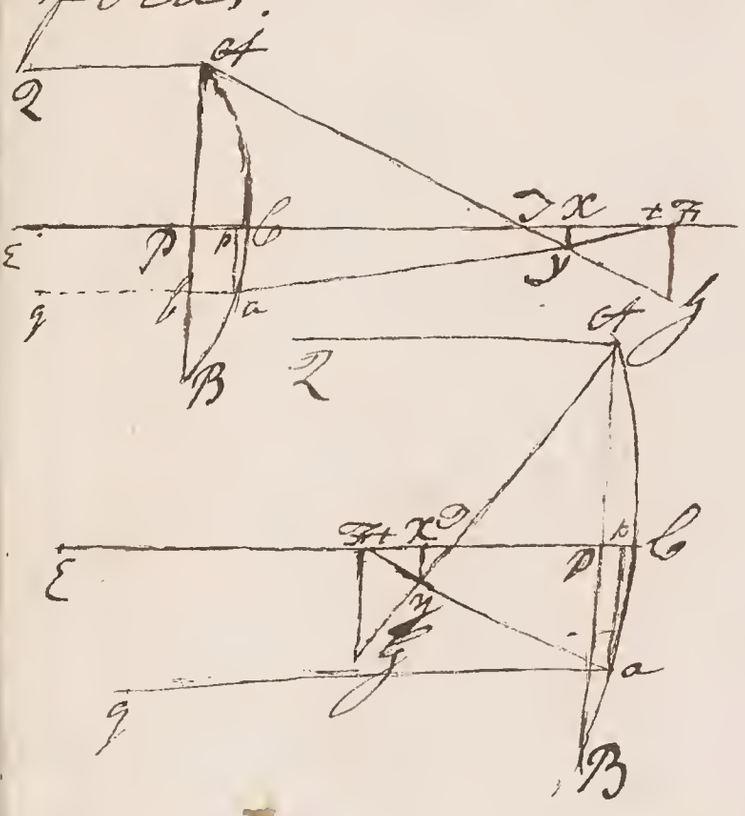
The lateral aberration $F f = \frac{A P^3}{2 E E^2}$, D or $F f : D F ::$ $:: A P : P D$ or $\frac{E E}{2}$.

When the diameter of the concave or its focal distance is given the longitudinal aberrations are as the squares and the lateral ones as the cubes of the diameter of the apertures

homogeneal

When parallel rays are refracted by a spherical convex lens or reflected from a spherical speculum, the diameter of the least circle of aberration caused by the curvature of the figure is a fourth part of the lateral aberration of the extreme rays at the principal focus.

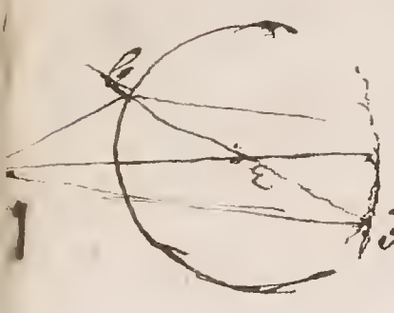
When parallel homogeneal rays are incident upon a spherical refracting surface or plano convex lens; or upon a spherical concave speculum; the radius of the least circle of aberration (caused by the sphericity of the figure, only) is a fourth part of the radius of the circle of aberration at the geometrical focus.



Let A C B be the spherical surface of a refracting medium, lens or speculum, whose geometrical focus, of rays parallel to the axis E C F, is F; let 2 A be the extreme ray of that pencil refracted or reflected in A G meeting the axis in Q and a perpendicular to the axis at Z in G; then it is evident that Z G will be the radius of the greatest circle of aberration at F the geometrical focus; Let g a be any other ray refracted or reflected in a G meeting the axis in t & A Z G in Y, from Y draw Y X at right angles to the axis; then, as Y is the intersection of the extreme ray with any other, when x Y is the greatest every refracted or reflected ray will pass through a circle described with x Y as a radius and it will be the radius of the least circle of aberration. Let $x = ZX$, $y = ap$ (drawn from a at right angles to the axis) $PZ = f$, $ZZ = b$, $AP = a$. Then as the radius E C of the sphere is given the longitudinal aberrations are as the apertures that is $a^2 : y^2 :: b : \frac{by^2}{a^2} = Xt$, and $Zt - Zx = \underline{Xt} = b - \frac{by^2}{a^2}$. Then, from similar triangles, $f : a :: x : \frac{ax}{f} = \underline{xy}$, $y : f :: \frac{ax}{f} : \frac{ax}{y} = Xt$, and $Xt + \underline{xx} = \underline{Zt} = x + \frac{ax}{y}$; therefore $x + \frac{ax}{y} = b - \frac{by^2}{a^2}$ or $x \times \frac{a+y}{y} = \frac{b}{a^2} \times a + y \times a - y$ or $x = \frac{b}{a^2} \times a - y \times y$, but $\frac{b}{a^2}$ is a given quantity therefore x will be a maximum when $a - y = y$ or when $y = \frac{a}{2}$, for y substitute $\frac{a}{2}$ in the last equation then $x = \frac{b}{a^2} \times \frac{a^2}{4} = \frac{b}{4}$ that is, when x or ZX or XY is a maximum ZX is $\frac{1}{4}$ of ZZ and, because XY & ZZ are parallel XP is $\frac{1}{4}$ of ZG but it has been proved that when XY was the greatest it was the radius of the

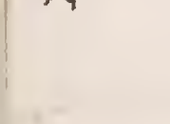
Therefore

To determine the point in which any single ray after refraction cuts the axis of a refracting spherical surface.



Let C be the point in which the ray 2 C, diverging from 2, is incident upon the spherical surface, upon C E; take a line 2 J which is to 2 C as the sine of incidence to the sine of refraction; from the center 2, with the radius 2 J, describe a circle meeting C E produced in J; 2 J is parallel to the refracted ray at C; for the sine of the angle 2 C J is to the sine of C J 2 as 2 J to 2 C that is as the sine of incidence to the sine of refraction; but 2 C J is the complement to the angle of incidence, therefore C J 2 is equal to the angle of refraction and consequently 2 J is parallel to the refracted ray.

When 2 C is to 2 E as the sine of refraction to the sine of incidence the ray will be refracted parallel to the axis, because then J coincides with E; When 2 C has a greater ratio to 2 E than R. to J the refracted ray will meet the axis on the same side of 2 with E because J is now on the contrary side of the axis that C is and 2 J makes an angle with it. When 2 C has to 2 E a less ratio than R to J the refracted ray meets the axis on the contrary side of 2 that E is because J is between C and E and 2 J makes an angle with axis on the same side with C.



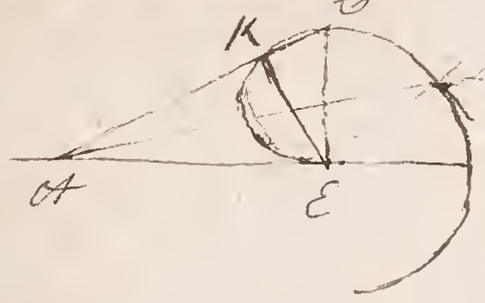
When it is at F, (the principal focus of rays coming in a contrary direction) all the rays very near the axis are refracted parallel to it, for it has been proved that A C is to A E as the sine of refraction to the sine of incidence.

To find that point in the axis which being made the focus of rays incident upon a spherical surface, the extreme rays or the tangents will be refracted parallel to the axis and the rest meet it on the contary side of it that E is,. Draw E K at right angles to the axis and take K E to E D as R to I; through K draw K C parallel to the axis meeting the surface in C; at C draw a tangent to the surface meeting D E in A which is the point req.

For the triangles K E C & A C E are simil. K E : E C or E D :: A C or E :: R : I.

When rays are incident upon the spherical surface of a rarer medium the directions of the refracted are found in the same manner as before; but AC in this case is greater than AJ or refraction is greater than incidence; when AC is to AE as refraction to incidence the ray is refracted parallel to the axis; when AC has a greater ratio to AE than K to J the ray meets the axis beyond E; when it has a less ratio the ray meets the axis beyond E.

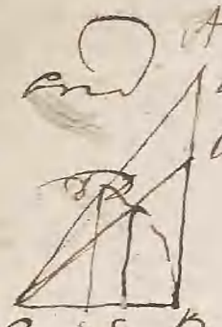
To find that point of incidence where extreme rays are refracted parallel to the axis; Through the center E draw EC at right angles to the axis meeting the surface in C; upon



EC describe a semicircle and take EK to EC as J to K join CK and produce it to L till it meets the axis in L which is the point required; For as the triangles ACE & ECK are similar AC : AE :: EC : EK that is as K to J; every other ray incident from A meets the axis beyond A as every other point L is at a greater distance from A than C therefore A L has a greater ratio to AE than K to J.

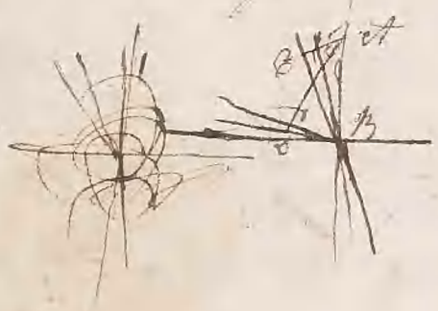
Let E be the center of a spherical surface of a dense medium; let C be any point in the surface join EC, take EA to EC as I to K as gE to EC as K to J; rays incident diverging from A will be refracted accurately diverging from g; For the the triangles ACE & CgE, having a common angle and their sides about it proportional, are similar; therefore Cg A is equal to the angle of incidence ACE; and the sine of CgA is to the sine of gCE as CE to gE that is as incidence to refraction; therefore gCE is universally the angle of refraction or its complement

Lemma 1st The Tangent of any angle is as its sine directly and its cosine inversely.

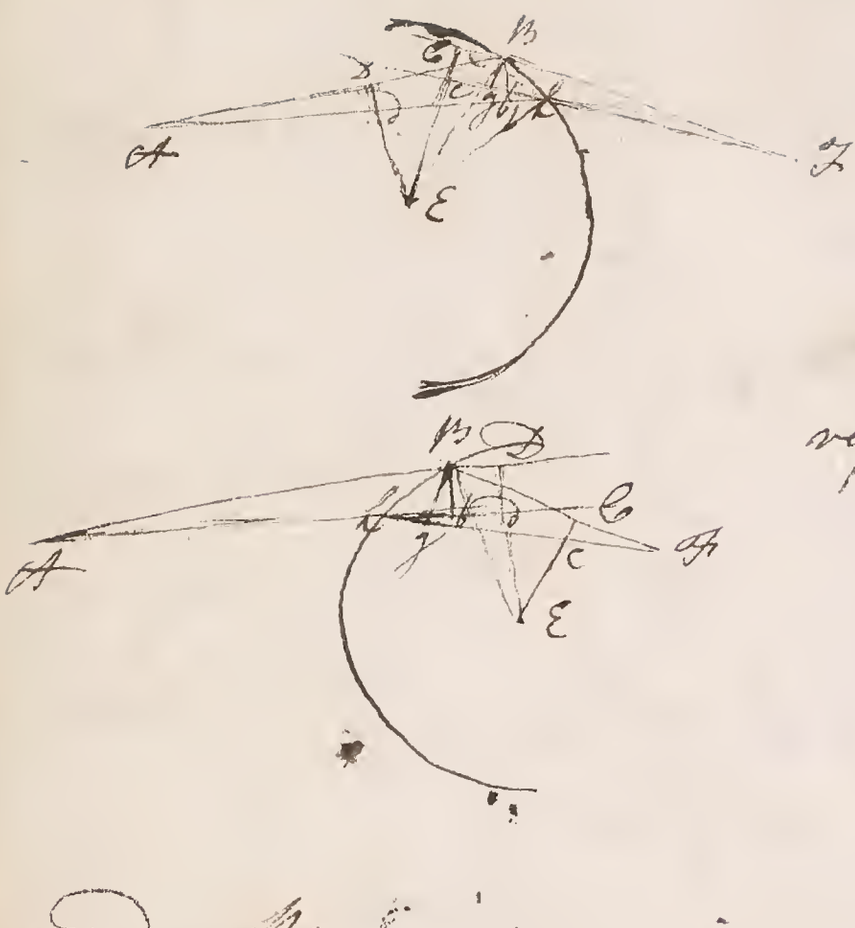


Let ∠ABD be any angle whose ~~tangent~~ is AD, its sine DE and let CB be radius: then as the triangles ABC & DEC are similar AB : BC :: DE : EC and AB = $\frac{BC \times DE}{EC}$ but BC is a given quantity, therefore AB : $\frac{DE}{EC}$ that is the tangent is as the sine directly and the cosine inversely.

Lemma 2 The increments or decrements of the angles of incidence and refraction are to one another as the tangents of those angles.

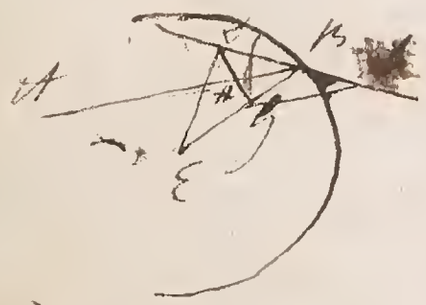


Let DB be a ray incident at B and refracted in BC Let Dd be increments of the angle of incidence and bc of the angle of refraction, and from any point of the perpendicular draw dD & dC at right angle to the incident and refracted rays; the angles are to one another as the arcs directly and radii inversely, DBd & CBd are to one another as the arcs Dd and Cc directly and the radii DB & CB inversely but Dd and Cc may be considered as the increments of the sines DA & CA and consequently they will be to one another as those sines; therefore DBd is to CBc as DA to CA directly and BD to BC inversely but DA & CA are the sines and DB & BC the cosines of incidence and refraction, therefore the increments or decrements are as the sines directly and cosines inversely, that is as the tangents of incidence and refraction.



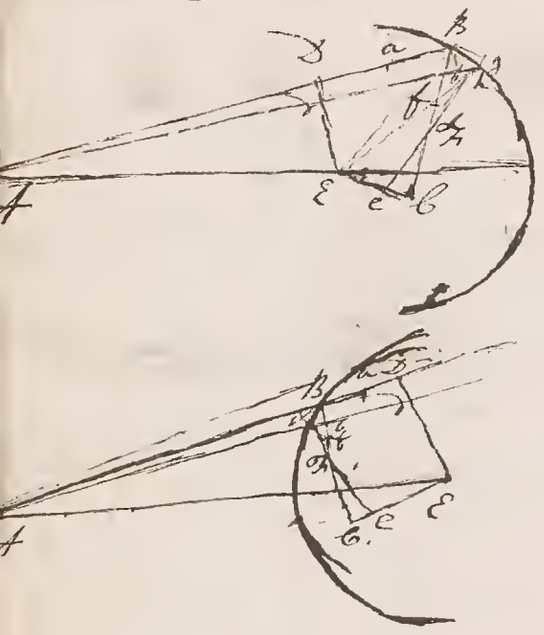
Let A be the focus of a small pencil of rays incident upon the spherical refracting surface whose center is E; let AB be the axis & ABC one of the extreme rays of this pencil refracted in BF & FH and meeting in F. From E draw ED at right angles to AB and EB at right angles to BF; then BG is to BC in the ratio compounded of the sines of incidence and refraction directly and the cosines inversely and of the distances AB & AD, corresponding with BG & FC. Let Bg & Bb be two small arc of circles described from the centers F & A with the radii BG & BC these may be considered as right lines perpendicular to their rad then as the triangles Bgh, BEE and Bbk, BDE are similar Bg : Bb :: Bb : BD and because Dd & Bb are parallel Bb : Dd :: AB : AD and Dd & Cc are the decrements of the sines of the angles DBE & dBE therefore Dd : Cc :: DE : CE therefore by compounding the ratios Bg to Cc that is FB to FC is in the ratio compounded of DE to EC & BC : BD and of AB to AD that is as the sines of incidence and refraction directly &c.

When the rays of the pencil are parallel, that is when A is removed to an infinite distance the ratio of AB to AD vanishes, and FB is to FC (as the sines directly and the cosines inversely, that is,) as the tangents of incidence to the tangent of refraction.



By this the focus after refraction of a small pencil of parallel rays may be found, for let AB be the incident ray & B the refracted one, from the center E draw ED at right angles to B (produced if necessary), from D draw Dg at right angles to EB, through g draw gF parallel to AB meeting B in F the focus required. For as Bg & AB are parallel FB : FD :: gB : DB that is as the tangent of incidence to the tangent of refra

A Caustic is the luminous figure formed by the intersections of rays reflected from the points of a spherical surface remote from its axis, and which do not all meet in the geometrical foci the points of which the curve is made up, are found by considering as the foci after reflection of very small pencils of rays falling remote from the axis and the ares of those pencils in every point a tangent to the curve

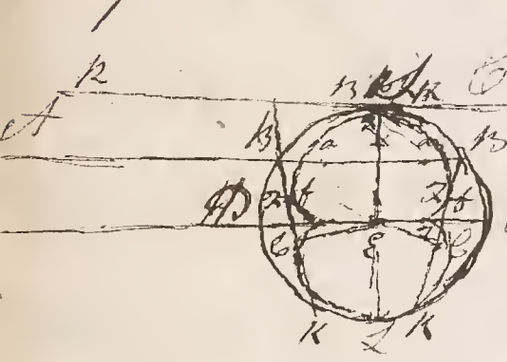


To determine the focus after reflection of a very small pencil of rays falling obliquely upon a spherical speculum.

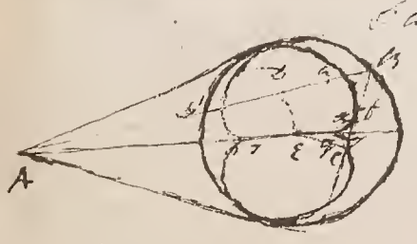
Let A B be the axis and A k any other ray of a very small pencil diverging from A and incident upon a circular speculum in the points B & k remote from the axis A E; let B C & ke be those rays after reflection; then, as A k is nearer to the focus than A B, ke will meet the axis in a point nearer to the center than B C therefore they must meet somewhere between the axis and the points of incidence; let F be that point.

Draw E D & E C at right angles to A B & B C and bisect D B & B C in a & f. A a will be to a B as B f to F f. Draw B b at right angles to A B meeting A k in b and B g at right angles to k e (produced if necessary) then A B : A D :: B b : D b (but D b & e b are equal as they are nearly the differences of the ares, respectively the angles D b B, A k E & C b B, E k e which are equal to one another, the angle A k e being nearly right angles, and because the triangles B b k & B g k are similar B b & B g are equal, and the triangles B F g & b F e are similar therefore B g : e b :: B F : F b) and A B : A D :: B F : F b and by composition and division $\frac{AB+AD}{2} : \frac{AB-AD}{2} :: \frac{BF+Fb}{2} : \frac{BF-Fb}{2}$; but A a & a B are two quantities whose sum is A B and difference A D, and B f & f F two quantities whose sum is B F and difference F b, therefore $\frac{AB+AD}{2}$ is equal to A a, $\frac{AB-AD}{2}$ to a B, $\frac{BF+Fb}{2}$ to B f and $\frac{BF-Fb}{2}$ to F f. Therefore A a : a B :: B f : f F.

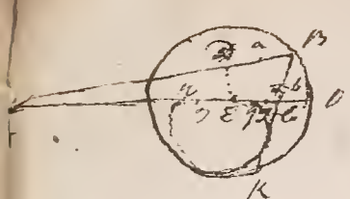
A caustic is a luminous curve which touches every ray after reflection of a pencil incident upon a spherical specalum. The points in which the curve touches any two reflected Rays whose points of incidence are very near to each other may be considered as coinciding with their point of intersection as the ~~line~~ perpendicular at that point is nearly common to both. Therefore the point D (as found in the last proposition) is always a point in the curve, And the Caustic is the locus of F.



Case 1st When rays are incident parallel BF is a fourth part of the chord BK; because as AB is infinite its ratio to to Ba will be infinitely great and Bf (which is finite) will have an infinitely great ratio to Bf; therefore D coincides with f and Bd becomes half of BC or a fourth part of BK; this caustic meets the specalum in the two extremities of the diameter drawn at right angles to the axis; because the rays RS incident at S evidently touches the sphere and as the angle RSE is a right angle it will be reflected in the same line, therefore the chord BK will vanish and D will coincide with S, the same may be proved of the other extremity. As AB is infinite, the real caustic formed by reflection from the concave surface SQL is equal to that formed by the convex surface SPL which is imaginary as the caustic is behind the surface by the reflection from which it is formed. The point D in which the ray passing through the center touches the caustic, is the geometrical principal focus for, as the incident ray passes through the center the reflected ray will pass through it and BK will become a diameter and CD a fourth part of it; which corresponds exactly with what has be demonstrated of rays falling very near the axis.

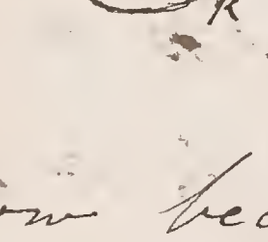


Case 2 When rays diverge from a point A, the locus of F will be found by the proportion Aa : aB :: Bf : fF. and the caust will meet the sphere universally in the points of contact of the tangents drawn drawn from the focus of incidence, as BK always vanishes at the points and the real caustic will be greater than the imaginary [illegible]



The point in which the caustic touches the rays passing through the center is the geometrical focus of rays after reflection which are incident diverging from A; because A B coincides with A O, D with E, a with O, b c with with O E and B f with O O, therefore 2 will coincide with 9 the geometrical focus.

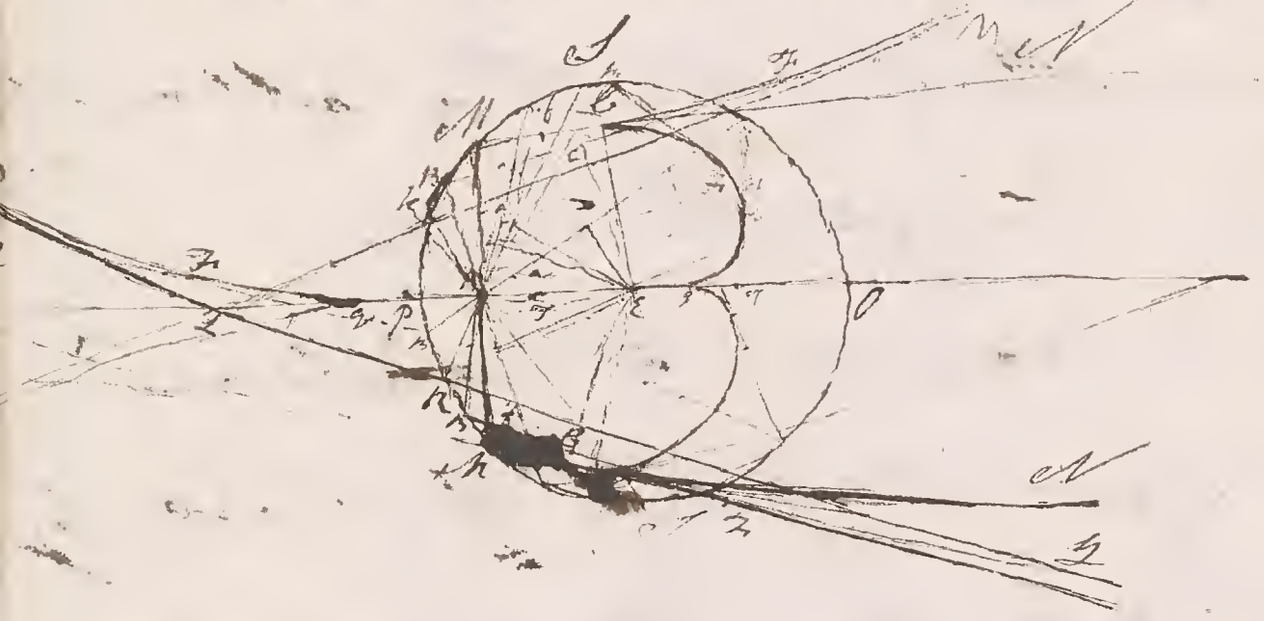
Case 3d. When the luminous point A is at the extremity of the axis



the imaginary point will vanish and the whole caustic will be real as it is reflected from the whole concave surface; B 2 is in this case a third of B K; for as A B now becomes a chord it is bisected in D and a B is a fourth part of A B therefore A B is $\frac{1}{4}$ a B and b f $\frac{1}{4}$ B f or $\frac{3}{4}$ $\frac{1}{4}$; but B f is a fourth of B K, therefore F f is $\frac{1}{3}$ of $\frac{1}{4}$ of B K or $\frac{1}{12}$ of B K and B f & f 2 together or B 2 is equal $\frac{1}{4} + \frac{1}{12}$ that is $\frac{1}{3}$ of B K.

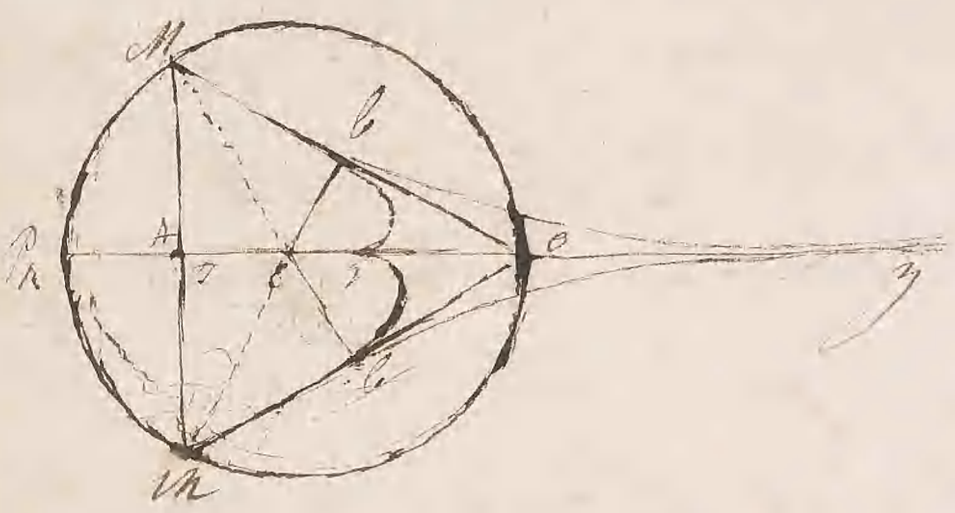
The point y in which the caustic touches the ray reflected in the diameter agrees as before with the geometrical focus; for, A O is to Z E or Z C. to 2 y as 3 to 1; therefore 2 y is $\frac{1}{12}$ of the diameter which being added to 2 O or fourth will give y O one third of A O which corresponds with the point of contact.

Case 4th. When the luminous point is between the surface and



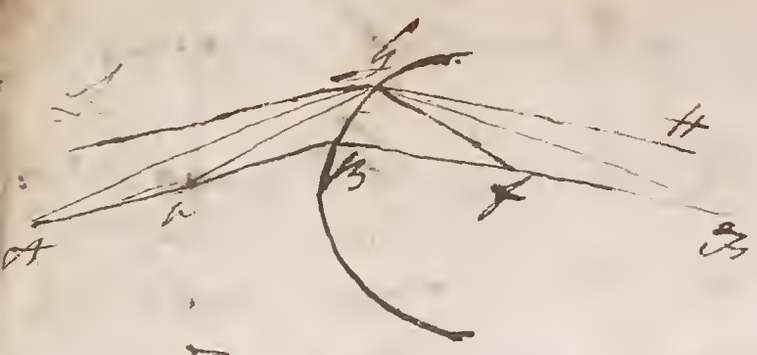
F; Let A R be that ray whose part A R is equal to a fourth part of the whole chord R I. the reflection from the surfaces B R & P R will form two curve infinite towards y whose asymptotes are the rays R y reflected from R and which touch the axis in a cusp at y which lays between L (the intersection of R y with the axis) or B the vertex; for in this case the point A is on the same side as that B is, therefore 2 must be on the contrary side of f than B is; the ray incident at B will touch in the geometrical focus y, because A D becomes A a. & the ray incident at R touches at an infinite distance, because E a vanishes. Let A M be be a ray at right angles to the axis reflected in M N; draw E b from the center as before; the caustic will touch M N in b because then (as A & D coincide) A a & a B consequently B f & f 2 are one equal; the part of the speculum R M forms a curve which touches N M in b and is infinite towards y with the asymptote R y: for the ray reflected from R touches at an infinite distance, and as B moves towards M 2 will come up towards b and will be on the contrary side A

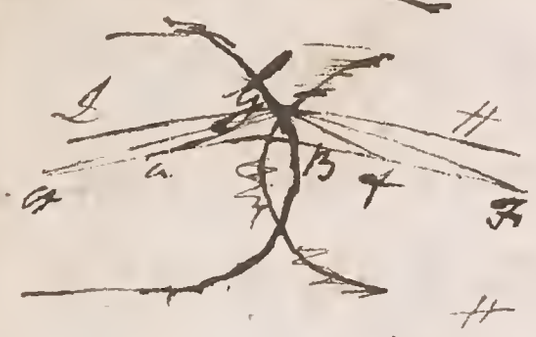
Case 5th When the luminous point is at D. Draw AM as before AM will meet the axis in the surface. Let E be the point in which MC meets the axis, join EM which will bisect the angle at M, therefore DE : EO :: AM : MC but EO is twice DE therefore MO is twice AM or MC which is half the chord therefore O is a point in the surface. As no line can be drown from the point A to the surface less than AD, the point between D & R and consequently the caustic formed by it will vanish the two asymptotes coincide with the diameter and the curves formed by MBM will touch at C as before and will be infinite towards y with the diameter produced as a common asymptote.

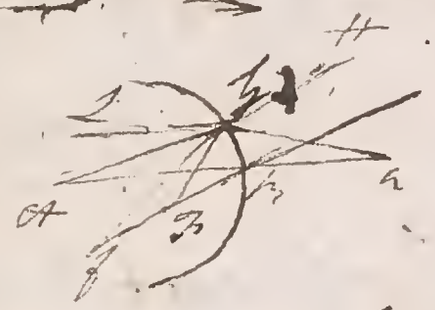


Case 6th When the luminous point is between D & E; AM meets the axis within the circle and the two curves which were before infinite now touch the diameter (if nicely produced) in a cusp. at the geometrical focus y. As the point A moves towards E the points y & C move nearer to one another till A coincides with E when they all meet, that is every reflected ray passes through the center.

In each th, that part of the caustic which is formed by the surface RBR is imaginary as it lies behind the curve by which it is formed: every other part of the caustic is real





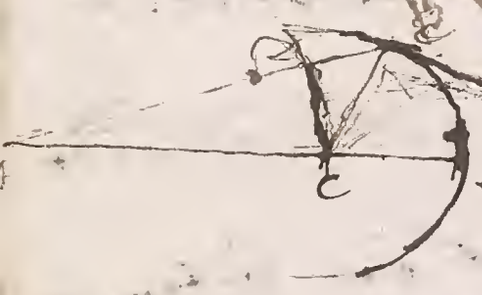


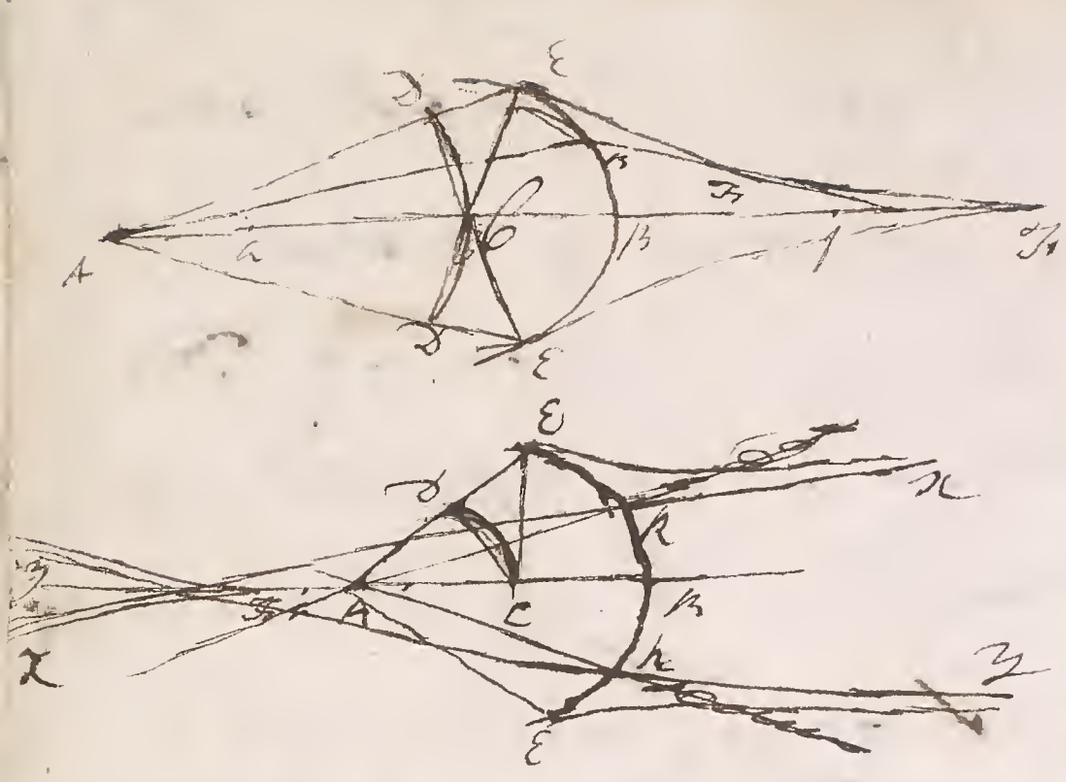
Let A B be the axis of a small pencil of oblique rays diverging from A whose focus after refraction is F; Let a be the focus after refraction of rays incident parallel to B F in a contrary direction and f of rays incident parallel to A B in the same direction; then will be to a B as B f to f F.

For, let G be a point in the surface very near to B, join A G, a G, f G & F G; then as the angles A G a A G f or G B f or their complements to two right angles are very small they are to one another in a given ratio (of the tangents) for the same reason A G a or G a f & f G F are in a given ratio; therefore A G a : G a f :: G B : B f. but these angle being very small are to one another as their sines that is as the sides of the triangles A G a & G f B to which they are opposite, therefore A a : A G :: G f : f B that is ultimately A a : a B :: B f : f F.

F will always be on the contrary side of f that A is of a and if A B be given in position A a & f F will be reciprocal as their rectangle is equal to a given quantity a B x B f.

To determine the points E between which all the rays incident diverging from a point A upon a refracting spherical surface and without which every ray will be reflected or, which is the same thing, to determine a point of incidence at which the refracted ray is at right angles with the diameter;



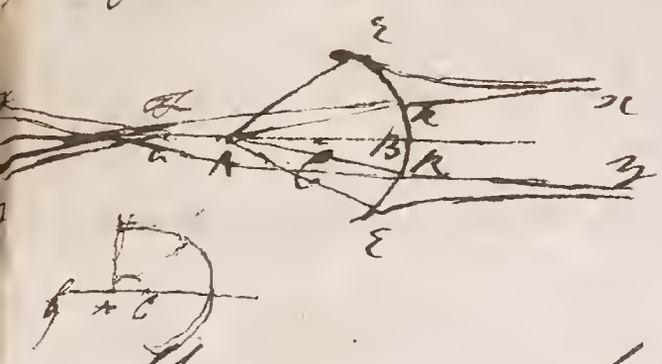


Let $\mathcal{E}B\mathcal{E}$ be a spherical refracting surface whose center is C, let rays diverge upon it from A in the dense medium; each ray cutting AC, the distance of A from the center, describe a semicircle and within it inscribe lines CD which are to CB as the sine of incidence to the sine of refraction; join AD and produce it till it meets the surface in $\mathcal{E}$ those points are those in which the caustic formed by the refraction at the surface $\mathcal{E}B\mathcal{E}$ touches the surface and no ray incident at a point more remote from the axis than $\mathcal{E}$ can form any part of the caustic by refraction.

Join $C\mathcal{E}$; then as $CD\mathcal{E}$ is a right angle CD is the sine of the angle of incidence $D\mathcal{E}C$ but DC is to $C\mathcal{E}$ (or CB) as the sine of incidence to the sine of refraction; therefore $C\mathcal{E}$ is which is the sine of $CD\mathcal{E}$ is. The sine of refraction and consequently the refracted ray at $\mathcal{E}$ is at right angles to $C\mathcal{E}$, therefore $\mathcal{E}$ coincides with $\mathcal{E}$ as with D (by the construction) and $\mathcal{E}$ vanishes, therefore F coincides with $\mathcal{E}$. From this it is evident that the caustic formed by refraction is only formed by the surface between the points $\mathcal{E}$, as every angle of refraction above $\mathcal{E}$ is greater than a right angle and consequently the ray is reflected

The two branches of the curve touch the surface in the points $\mathcal{E}$ and meet in the focus of the geometrical focus is if AC be greater than a C; but is AC coincide with a, as the geometrical focus is then at an infinite distance the curves will be infinite with the axis as an asymptote. When AC is between a & C, let AK & AK be the two rays in which a coincides with AC; the caustic formed by the surfaces $\mathcal{E}K$ & $\mathcal{E}K$ will touch the surface as be in $\mathcal{E}$ and will be infinite with the refracted rays

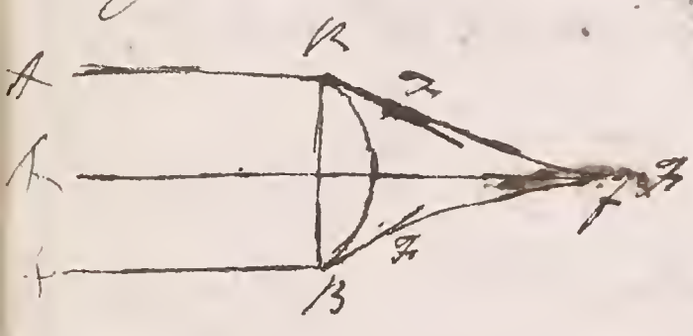
asymptotes as F is in that case at an infinite distance.



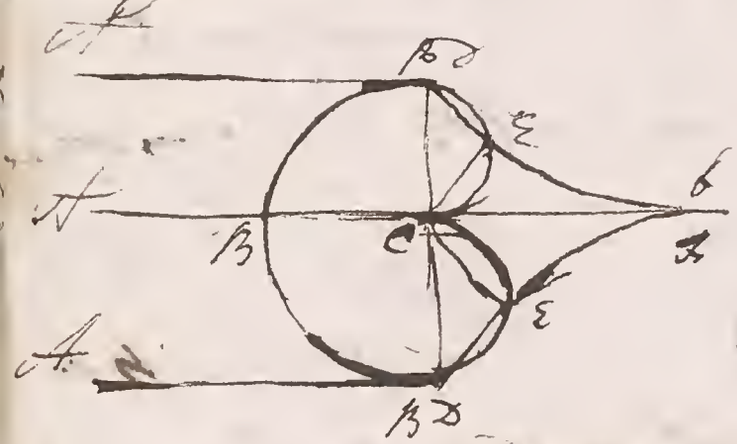
These same refracted rays produced are asymptotes to the other caustics which by axis meet in a cusp at the geometrical F.

They are formed by refraction at the surfaces R B, R B and as they are behind those surfaces they are imaginary.

When A C is to C B as C R to R F the whole caustic commences with the imaginary point F whose distance F C is to C B as R to F; for it has been proved before that every rays will accurately diverge from that point

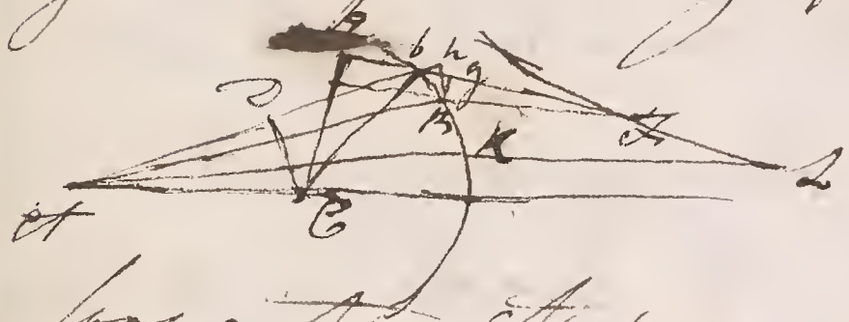


If parallel rays are incident perpendicularly upon the plane side of a plano-convex lens the caustic may be found as in the last case only that it will not touch the surface unless the spherical part of the lens reaches as far as the points E; the caustic begins at the points B of the rays refracted at the extreme points B.

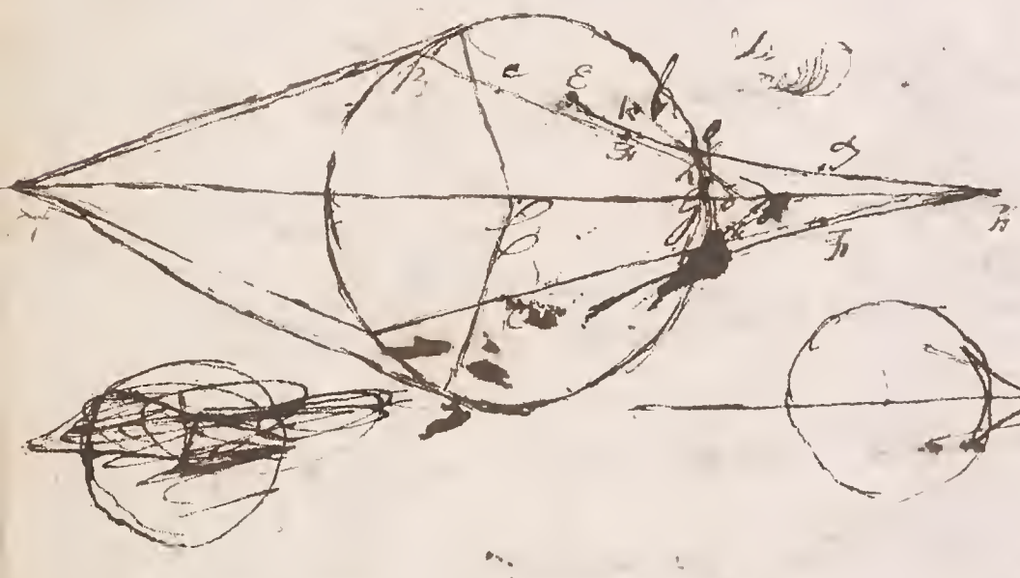


Let D B D be the convex surface of a dense medium; let A D and A D be the tangents, join C D, C D and upon these radii describe the semicircles D E C; inscribe C E and let it be to C B as K to D; the caustic begins at the point E and meets the axis at the geometrical focus F if A C is greater than a C but if not the caustics will seperate as in the first case; and part of it will become imaginary as before.

To find the length of any part of the caustic by a refracting spherical surface



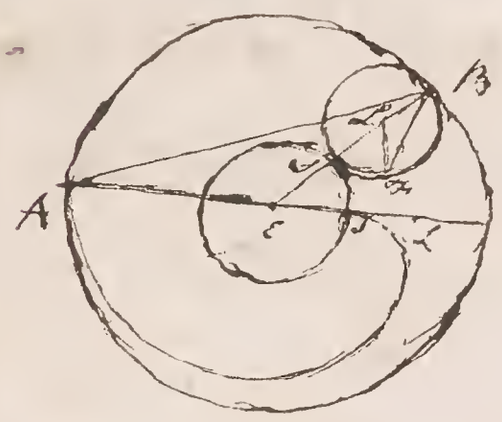
Let DL be part of a caustic formed by the surface BK let BD be the tangent at one extremity and KL to the other; then as the small increment of AB (bh) is to the small decrement of BD (bg) in the given ratio of DC to CE that is of D to B (the figures bhg BDE being similar) the sum of those increments or AK − AB is to the sum of the decrements or BD + DL − KL as D to r; therefore BD + DL − KL is equal to AK − AB × $\frac{D}{r}$ or DL = AK − AB × $\frac{D}{r}$ + KL − BD.



To find the caustic formed by the refraction of rays at both the surfaces of a complete circle (the form of a cylinder or the great circle of a sphere)

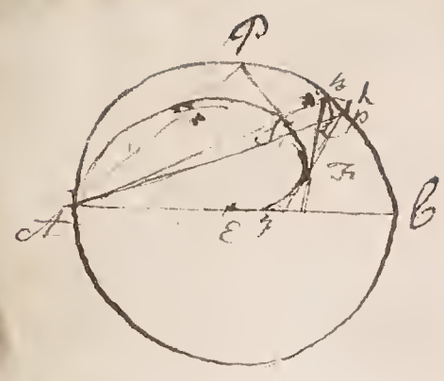
Let A be the focus of incidence of rays diverging upon the circle; let EBF be the caustic formed by the refraction at the first surface only, as in the last case; let AB be ray refracted at the first surface in BL touching the caustic in F; let it be refracted at the second surface in LD; let D be the principal focus of rays incident in BL and C the principal focus of those in DL; then as the small pencil incident in AB has a focus at D it will be incident upon the second surface with D as its focus of incidence that is, if F is within the circle the rays will diverge and if without they will converge upon the surface DL; therefore the second focus of the pencil will be found by the proportion Fc : ch :: LD : Dk; When F is between C & L, k is always within the circle for LD is then less than Dk, when F is beyond L, k is always without the circle, because then Dk is greater than Dk; therefore the part of the second caustic corresponding with EBO is entirely within the circle & F is every where between C & L and the part corresponding with OBF is entirely without the circle; The second caustic passes through the point O in which the first caustic cuts the circle, for when the focus of incidence is in the circle it universally coincides with the focus of refraction. The second caustic will meet the circle again in a point T, where the ray is a tangent as r in that case coincides with L.

If one circle be made to revolve about another of an equal radius, the motion of any one point in that circle will form the caustic in the third case, in one complete revolution.



Let ABC be a circular speculum whose axis is CAC; let the radius CC be divided into three equal parts, D in the points I & L; from C as a center describe a circle with the radius CI, and from the center L describe a circle with an equal radius LI or LB; let the circle whose center is C remain fixed and let the other revolve about it; the motion of the point I in a complete revolution will describe the caustic formed by rays diverging from the luminous point placed at A. Let AB be one of the rays incident at B, join CB; then as the circle L has moved along the arc IB the point I will be at B and the arcs BI & DB will be equal and, as the circles have equal radii the angles DLI & DCB and consequently their halves ABC & LBI will be equal to one another; therefore the reflected ray passes through B; and, as the triangles ABC & DBL are similar, AB into BI (as CB to BL that is) as 3 to 1, therefore I is a point in the caustic and the same may be proved of any other point D. Therefore the locus of I is the caustic.

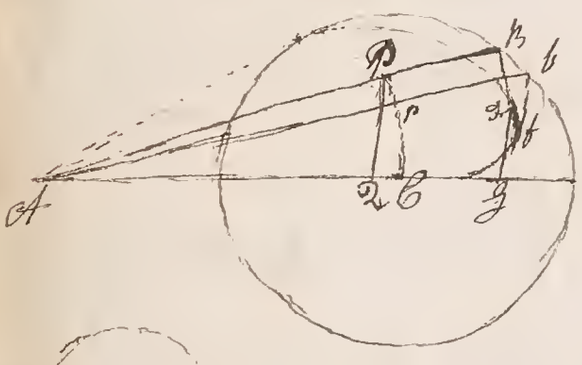
The length of any part of the caustic is equal to the difference between the sums of the incident and reflected rays at its extremities. If a string be fastened at one extremity to the point A and at the other to the point Z; let part of the string lap round part of the caustic ZI and let it be stretched at B so that AB and BI be straight lines; let B be always a point in the speculum; AB+BI+IZ, the whole length of the string, is a given quantity.



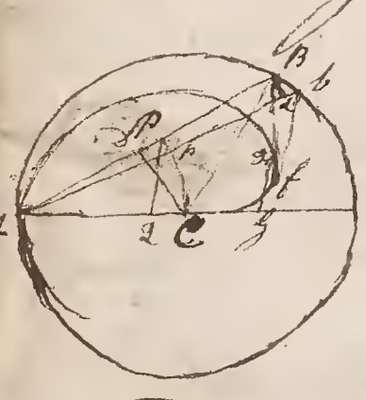
because $\angle P$ the increment of of AB is equal to Ph the decrement of BZq, the triangle Bhp & BLp being equal

The length of any part CZ of the caustic is equal to the difference between $AP + P\pi + \pi\rho$ (or) $AZ + Z\rho$ and $AB + B\beta + \beta\rho$ for, taking away the common part $Z\rho$, the remainder $AP + P\pi + \pi\beta$ will be equal to $AB + B\beta$ and CZ to $AB + B\beta - AP + P\pi$.

From this it is evident that the length of the whole caustic on one side of the diameter is equal to $AC + Cq$ or $\frac{4}{3}$ of the diameter therefore the whole caustic will be equal to $\frac{8}{3}$ of the diameter or $2\frac{2}{3}$.

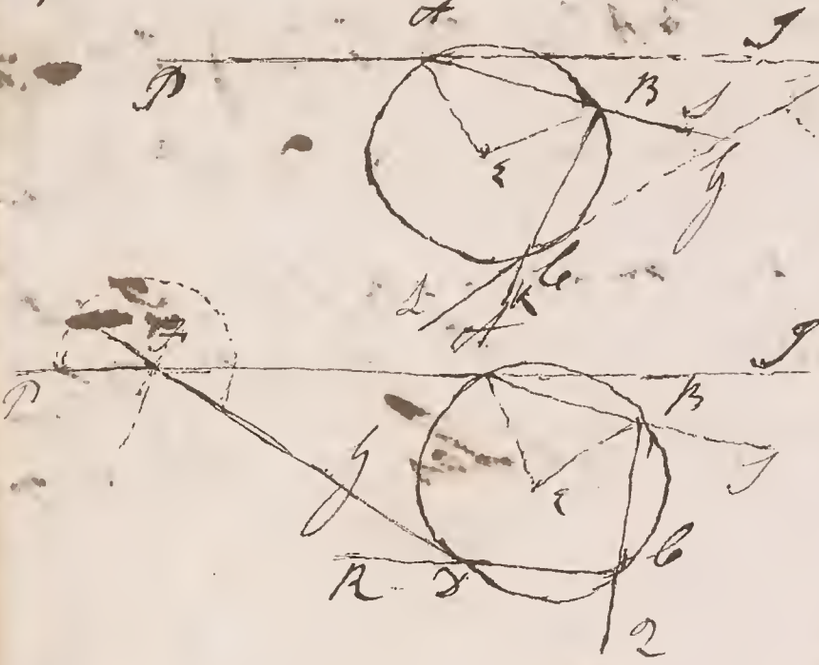


To determine the density of the light at any point of the caustic; from the center C describe a circle passing through C and meeting the small pencil of rays incident upon the sphere at Bb; let Zf be part of the caustic formed by the pencil incident at Bb; draw $P2$ & Zq at right angles to the axis; then as there are the same number of rays between the points P & p before that there are between Z & f after reflection the density of the light at the point P & Z will be as the spaces Pp & Zf inversely or supposing the rays to be incident upon a sphere; the density in Zf will be to that in $P2$ as $Pp \times P2$ to $Zf \times Zq$ or, because Pp is a given quantity the density in Z will be as $P2$ directly and $Zf \times Zq$ inversely.



In the case in which A is at the surface; draw CD at right angles to AB join B, C and let BD be a small arc of a circle described with the radius AB, which may be supposed to be a right line perpendicular to AB & Cb; then, as the triangles ABC, APp and CDB, CPB are similar, $Pp : BD :: AP : (AB$ or$)\ 2BC$ and $BD : Bb :: BX : XC$; because $Ab + bf - AB + Z$ or $\frac{4}{3}Ab - \frac{4}{3}AB = \frac{4}{3}Bb$ is equal to Zf, $Bb : Zf :: 3 : 4$; therefore, by compounding the ratios $Pp : Zf :: 3AP : 8BC$; but the densities at P & B are inversely as the surfaces generated by Pp & Zf; therefore the density at Z is to the density at B as $3AP \times P2$ to $8BC \times Zq$ or, because $P2$, BC and AP, AC are equal, as $3AC$ to $8AC$ to 8.

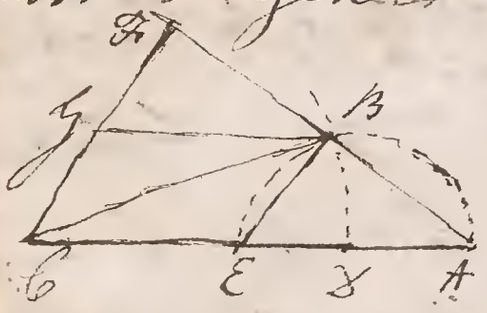
If a ray of light impinge upon a refracting sphere, and emerge after having been refracted into the sphere at the point of incidence and reflected at the interior surface a certain number of times, half the angle contained between the incident and emergent rays is equal to $n + 1 \times r - i$; putting i & r for the angles of incidence and refraction.



Let P A be a ray incident at A upon a sphere A B C whose centre is E; let it be refracted to B where it may either emerge or be reflected to C at which point it may emerge or be again reflected to D and so on. Produce P A, A B, B C &c, then it is evident that the whole deviation of the ray from its first direction P A is equal to I A B + I B C + 2 C D + ∠ C K = I A B that is, (as I B C, 2 C D ∠ C are equal to $n \times$ I B C + 2 I A B; join E A ∠ E K. Then as B is a point of reflection the angles A B E & E B C are equal to one another and to E A B that is A B C is equal to 2 E A B or 2 r but ∠ B C is the complement of A B C to two right angles Therefore ∠ B C = 2 R − 2 r (putting R for a right angle) and $n \times$ I B C = 2 n R − 2 n r; as I A B (putting R for a right angle) and $n \times$ I B C = 2 n R − 2 n r; as I A B is the difference between I A E & B A E, 2 I A B = 2 i − 2 r; therefore the whole Deviation or $n \times$ I B C + 2 I A B = 2 n R − 2 n r + 2 i − 2 r.

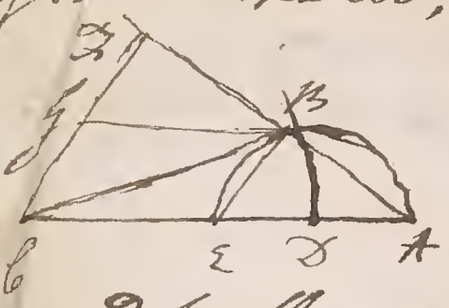
Then, as the angle between the incident and emergent rays is evidently equal to $n \times$ 2 R − the whole deviation or 2 n R − 2 n r + 2 i − 2 r = 2 n r − 2 i + 2 r = $2 n + 2 \times r - 2 i$, half the angle contained between the incident and emergent rays is equal to $n + 1 \times r - i$.

To construct two angles whose sines shall be in one given ratio and their tangents in another given ratio.



Let C A be any line; take a part of it, C D, so that C A and C D shall be in the given ratio of the sines; take another part C E and let C A and C E be in the given ratio of the tangents; upon E A describe a semicircle and from the centre C with the radius C D describe another circle meeting the first in B; join A B and produce it till it meets C F at right angles; join C B and through B draw B G parallel to C A; C B F and G B F are the angles required. Join B E, it is evidently parallel to F C; Then the sine of C B F is to the sine of G B F as C A to C B or C D (because G B & C A are parallel and C B A is the complement to C B F) that is in the

given ratio; Again, as $\beta\varepsilon$ is parallel to δc, $A\zeta$ is to $\zeta\varepsilon$ (as δA to $\delta\beta$ or) as $\delta\zeta$ to $\delta\eta$ that is as the tangents of $\zeta\beta\delta$ to $\eta\beta\delta$ tangent of $\eta\beta\delta$.

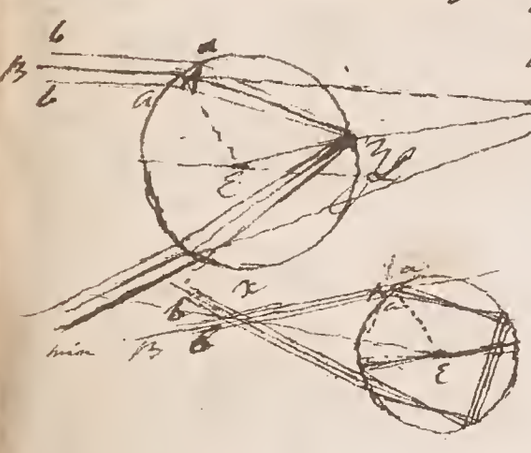


If the sines be taken in the ratio of I to R and the tangents of $n+1$ to 1 $c\beta$ will be to $\beta\delta$ as $\sqrt{n+1^2-1\times R^2}$ to $\sqrt{I^2-R^2}$. For, put $m=n+1$, then $cA : c\beta :: I : R$ and $\delta A : \delta\beta :: m : 1$. and $cA^2 = \frac{I^2}{R^2}\times c\beta^2$ and $\delta A^2 = m^2\times\delta\beta^2$ therefore $\frac{I^2}{R^2}c\beta - m^2\delta\beta^2 = \left(cA^2 - \delta A^2 = \delta c^2\right) c\beta^2 - \delta\beta^2$ or $\frac{I^2}{R^2}c\beta^2 - c\beta^2 = m^2\delta\beta^2 - \delta\beta^2$ or $\frac{I^2}{R^2}-1\times c\beta^2 = m^2-1\times\delta\beta^2$ therefore $c\beta^2 : \beta\delta^2 :: m^2-1 : \frac{I^2}{R^2}-1$ or $c\beta : \beta\delta :: \sqrt{n+1^2-1}\times R^2 : I^2-R^2$

But if $\beta\delta$ be made radius $\beta\delta$ will be the sine of the angle $\delta c\beta$ or the cosine of $c\beta\delta$ which may be found by the tables as the proportion consists of given quantities.

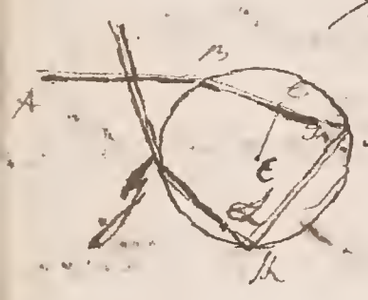
Half the angle contained between the incident and emergent rays is the greatest possible or its complement to two right angle the least possible when the tangent of incidence is to the tangent of refraction as $n+1$ to 1. For the quantity $n+1\times k-1$ is a maximum when the increment of k is to the increment of I as 1 to $n+1$ that is when the tangent of incidence is to the tangent of refraction as $n+1$ to 1 therefore half the angle &c.

If the incident ray contains the greatest possible angle with its emergent ray those rays which impinge at a very small distance on all sides of it will emerge parallel.

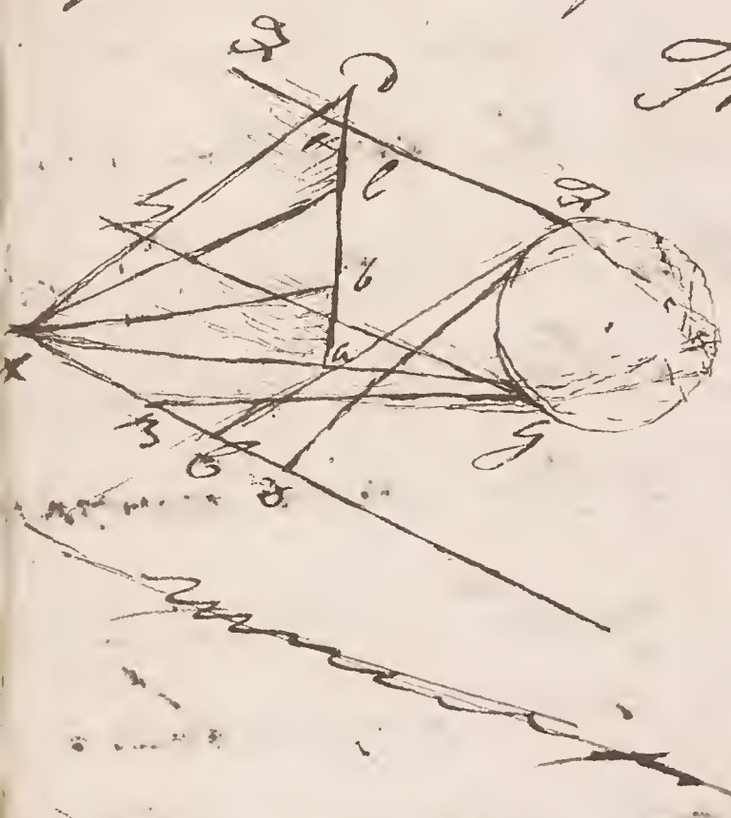


Let the angle of incidence $\chi\beta\varepsilon$ of a ray $\beta\chi$ be equal to $c\beta\delta$ as found by the construction, then as $\chi\beta$ is to sine $\zeta\beta\delta$ as I to R $\zeta\beta\delta$ will be its angle of refraction and as the tangents of those angles are as $n+1$ to 1, the angle $\beta\chi\varepsilon$ or half the angle between the incident and emergent rays will be a maximum; Let $b\alpha$ be ray parallel to $\beta\chi$ and very near to it; but it is evident that the difference between the angles of incidence and refraction α & χ are the increments of those angles therefore the are as the tangents or as $n+1$ to 1 and the increment of the angle $\beta\chi\varepsilon$ is nothing or $b\alpha$ makes the same angle with its emergent ray that $\beta\chi$ does with its corresponding emergent ray that is $b\alpha$ emerge parallel. 2 D

This last proposition may be demonstrated by supposing AB the axis of a small oblique pencil of parallel rays. Viz. let BZ be the direction of the axis after refraction; then if any point Z in the focus after refraction it has been proved that BZ will be to BC as the tangent of ~~incidence~~ refraction to the tangent of incidence that is (when the angle is a maximum) as $1 : n+1$. Viz. let $n = 1$ then BZ will be double BC and consequently Z will be in the surface of the sphere; then as the pencil is to be reflected at the second surface Z will become the focus of incidence upon the third surface at G. and as the chords BZ & ZG are equal it is evident that the pencil will emerge parallel at G.

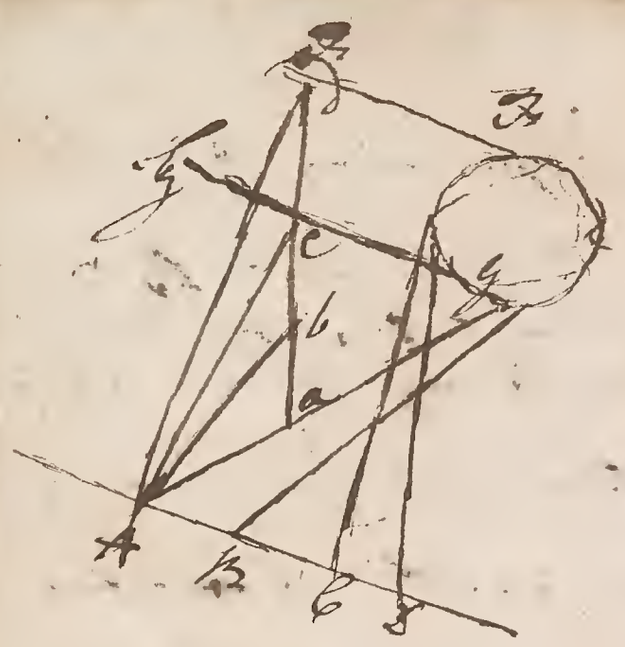


If there are two reflections BJ will be to BC as 3 to 1 that is BZ will become equal to $\frac{3}{4}$ and consequently ZJ to $\frac{1}{4}$ of BJ, therefore if Z be again made the focus of incidence the pencil will be reflected from D parallel upon K where it will converge to L so that KL shall be a fourth part of KG or of BJ and LG = BZ therefore if L be the focus of incidence the pencil will emerge parallel at G. In the same manner it may be proved that they will emerge parallel after any number of reflections.

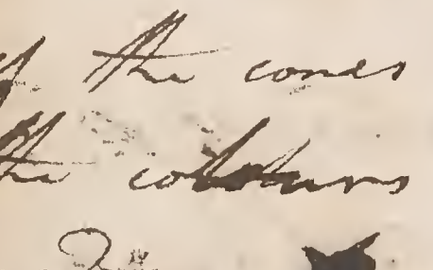


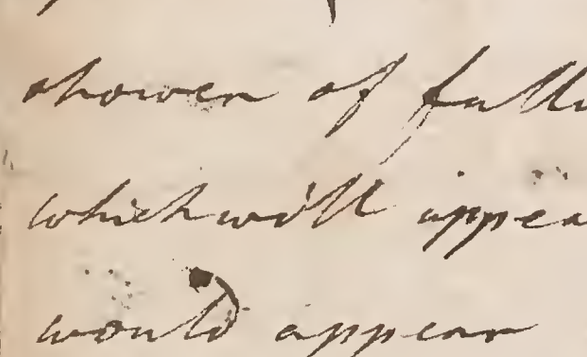
The ~~Rainbow~~ of the Rainbow.

Let ZG be drop of rain exposed to the rays of the sun; let ZB be that ray which has its angle of incidence with its emergent ray after one reflection a minimum and let GG, on the contrary side, have the angle of incidence with its emergent ray, after two reflections, a minimum; let AB be a line meeting the emergent rays; then as different colours have different degrees of refrangibility every separate colour will have a maximum or minimum of its own; Let A be the point in which the most refrangible or violet pencil meets AB after one reflection and B the point in which the red meets it; as AB is parallel to the incident rays the angles at A & B are the respective maxima and it is evidently less than B; Let C & D be the points in which the red & violet rays meet AB after two reflections; then the angle at C will be greater than



than the angle at D. As the number of particles and the ratio of the sine of incidence to the sine of refraction of the different colours are given the angle at A, B, C & D may be found by the tables; A = 40°17', B = 42°2', C = 50°57', D = 54°7'. The angles which any of the intermediate colours make with AB may be found in the same manner. Now suppose an eye placed at A it is evident that it will there receive a pencil of the most refrangible or violet rays if it be moved from A to B it will receive at B a pencil of red rays, and as it passes from A to B it will see all the intermediate colours in their order when it comes to C it will see red and at D violet as before; Now suppose the eye first at A, and through A draw lines AE, AC, AD making the angles EAD, CAD & DAD equal to those at B, C, & D; Then if a drop of rain be placed any where in each of those lines as at a, b, c, & D the eye will receive two pencils of red and two of violet rays or if a drop be placed anywhere between a & b or c & d the eye will receive a pencil of one of the intermediate colours, therefore if the whole space between a & d be filled with drops the eye will see two sets of colours in which the red will be the nearest & the space between them and the violet on the extremities. Let AD be made the common axis of the cones whose sides are Aa, AC, Ab & the directions of all the colours after refraction, and let the whole medium in which this cone is described be filled with drops of rain (which is the case when the sun shines upon a cloud or shower of falling rain) then the eye will see two circles of colours which will appear concentric to each other; the violet of each colour would appear but a line were the sun a perfect point, but he subtends an angle of about half a degree therefore the rays will be reflected from many more drops consequently the breadth of the bows will be more than about half a degree and the colours will be mixed with one another.





It is impossible to see the circle complete as a great part of it is covered by the horizon. The apparent altitude of the highest point of either of the bows is measured by the difference between its apparent diameter and the sun's elevation.

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